SPIE Optical System Design 2018, Paper 10694-24

Semi-analytical Fourier transform and its application to physical-optics modelling

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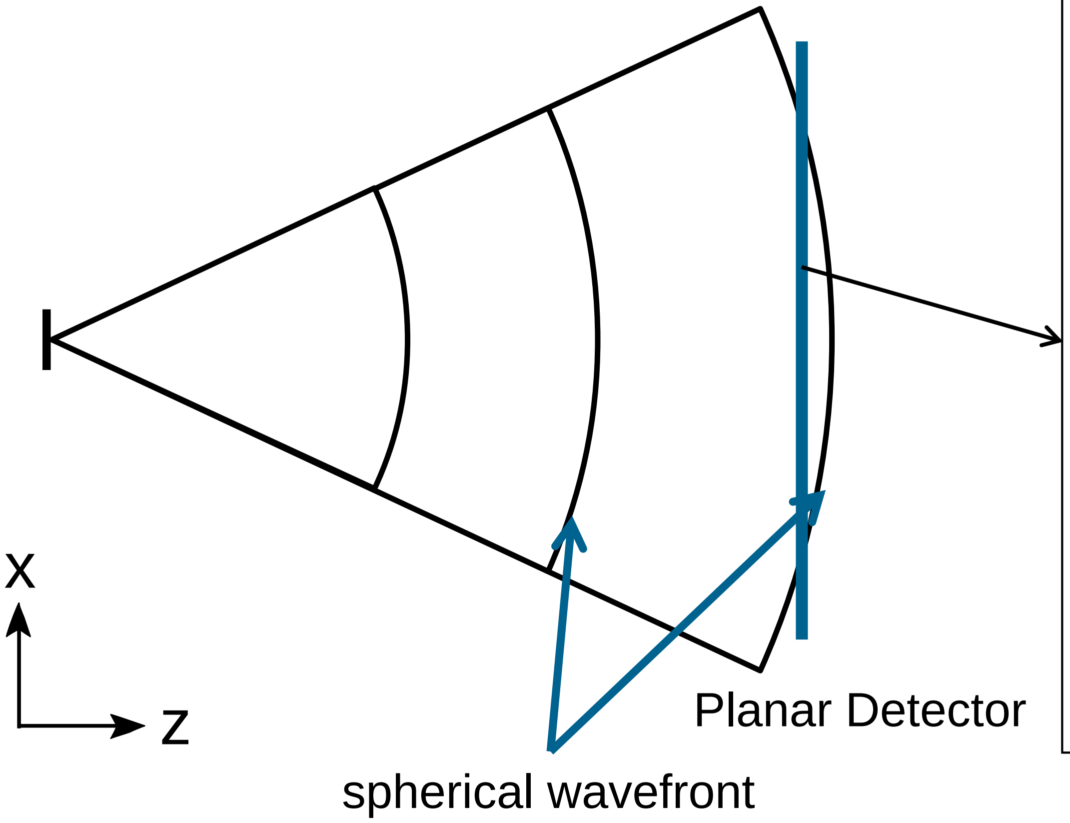
Introduction

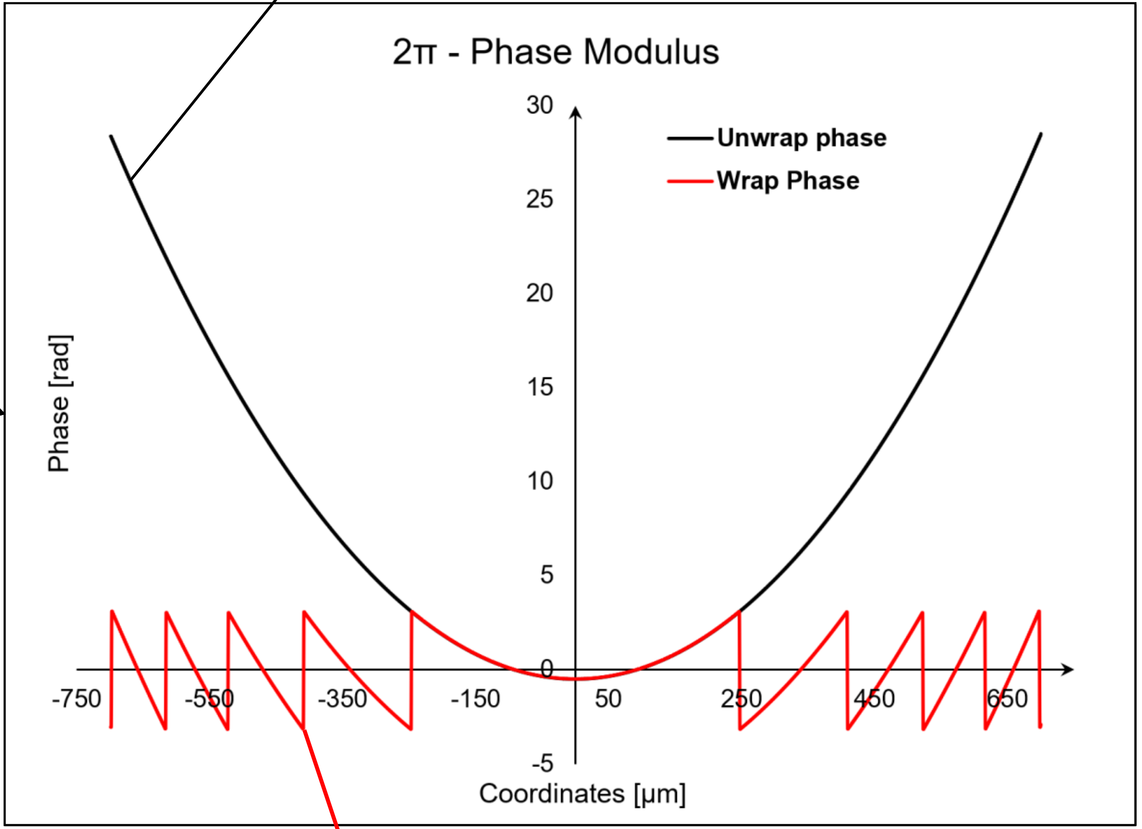
- Fast Fourier transform (FFT) in physical optics modeling and design.
  - Physical meaning: plane wave decomposition
  - Angular spectrum based algorithms: SPW, plane interface and etc.
- In practice, field with **strong wavefront phase**
  - huge numerical effort
  - 2π – Modulus of phase: range of phase $[-\pi, \pi]$
  - wrapped phase
  - Nyquist theorem ➔ sampling effort.
- Semi-analytical Fourier transform (SAFT)
  - **rigorous** approach
  - analytical handling of **quadratic phase**

Wavefront Phase of Electromagnetic Fields

- Wavefront (equal phase surface) in far field region

$\psi(\boldsymbol{\rho}) = \sqrt{x^2 + y^2 + r^2}$ spherical phase





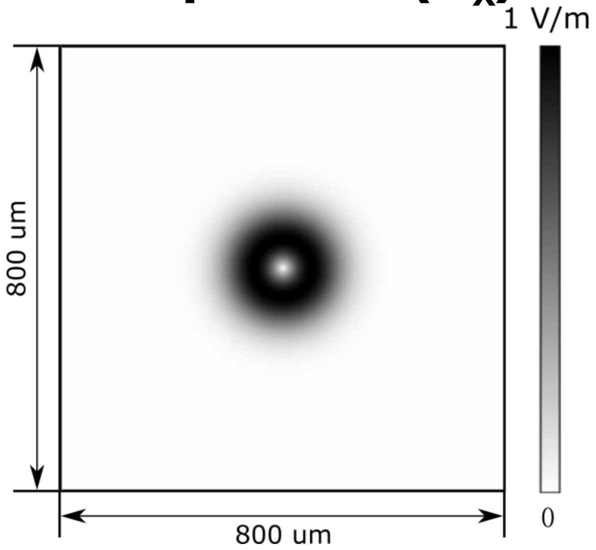
$e^{\mathrm{i}\psi(\boldsymbol{\rho})} = \cos\psi(\boldsymbol{\rho}) + \mathrm{i}\sin\psi(\boldsymbol{\rho})$ Nyquist Theorem

Example: Laguerre Gaussian 01 Mode in Far Field Zone

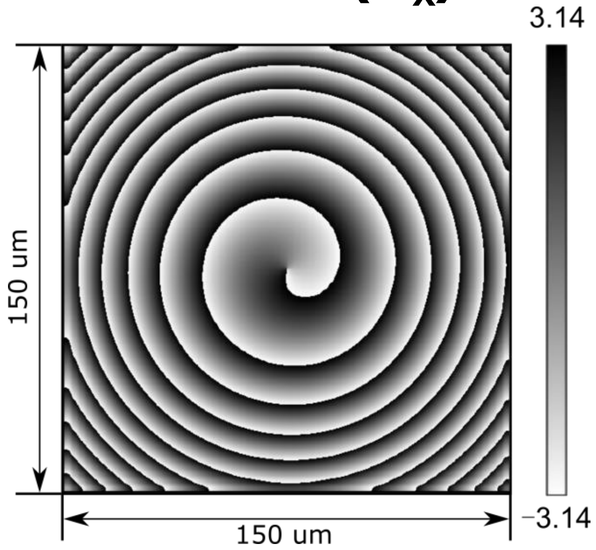
$$V_\ell(\boldsymbol{\rho}) = |V_\ell(\boldsymbol{\rho})|e^{\mathrm{i}\gamma_\ell(\boldsymbol{\rho})}$$

$V = (E, H)$ and $\ell = 1, \cdots, 6$ indicates the six field components.

Amplitude (E_x)



Phase (E_x)



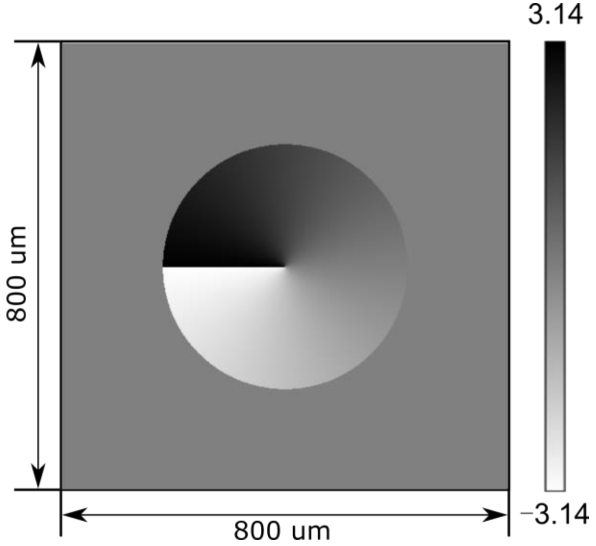
Wavefront Phase

$$V_\ell(\boldsymbol{\rho}) = U_\ell(\boldsymbol{\rho})e^{\mathrm{i}\psi(\boldsymbol{\rho})}$$

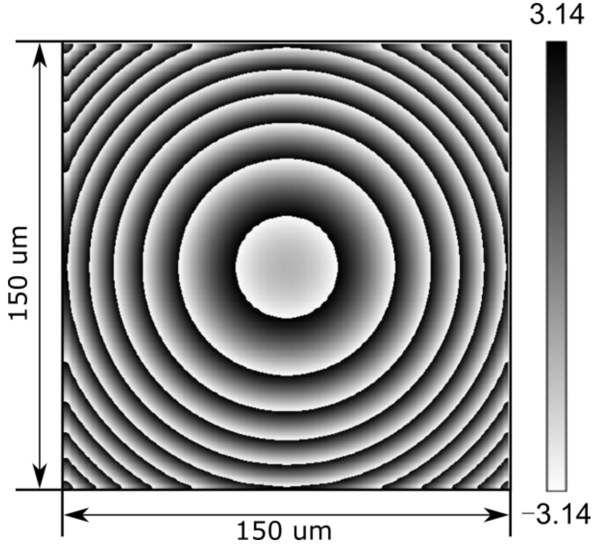
$$= |V_\ell(\boldsymbol{\rho})|e^{\mathrm{i}\varphi_\ell(\boldsymbol{\rho})}e^{\mathrm{i}\psi(\boldsymbol{\rho})}$$

$$\psi(\boldsymbol{\rho}) = \sqrt{x^2 + y^2 + r^2}$$

Diffractive phase



Wavefront phase



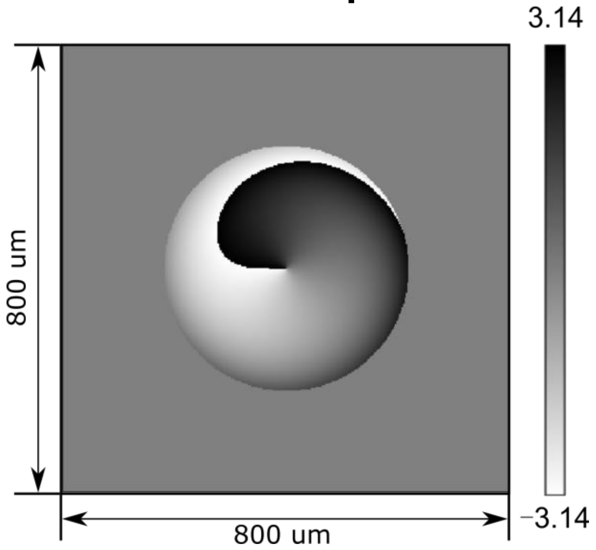
Quadratic Phase

$$V_\ell(\boldsymbol{\rho}) = U_\ell^{\mathrm{res}}(\boldsymbol{\rho})e^{\mathrm{i}\psi_{\mathrm{q}}(\boldsymbol{\rho})}$$

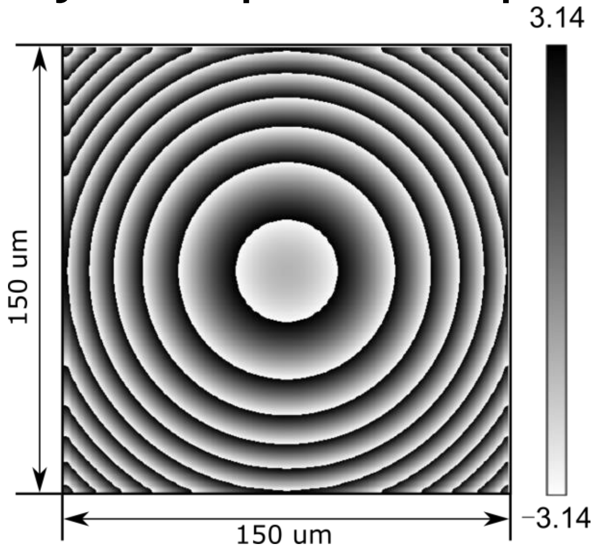
$$= |V_\ell(\boldsymbol{\rho})|\underbrace{e^{\mathrm{i}[\varphi_\ell(\boldsymbol{\rho})+\psi^{\mathrm{res}}(\boldsymbol{\rho})]}}\underbrace{e^{\mathrm{i}\psi_{\mathrm{q}}(\boldsymbol{\rho})}}$$

$$\psi_{\mathrm{q}}(\boldsymbol{\rho}) = D_x x^2 + Cxy + D_y y^2$$

Residual phase



Analytical quadratic phase

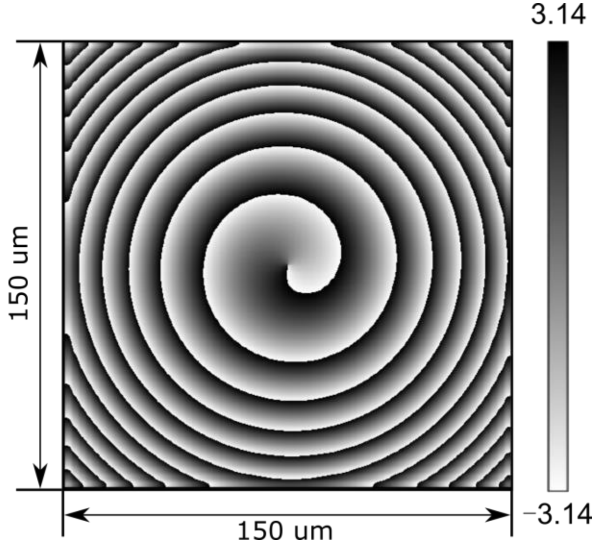


Analytical Handling of Quadratic Phase

$$V_\ell(\boldsymbol{\rho}) = U_\ell^{\mathrm{res}}(\boldsymbol{\rho})e^{\mathrm{i}\psi_{\mathrm{q}}(\boldsymbol{\rho})}$$

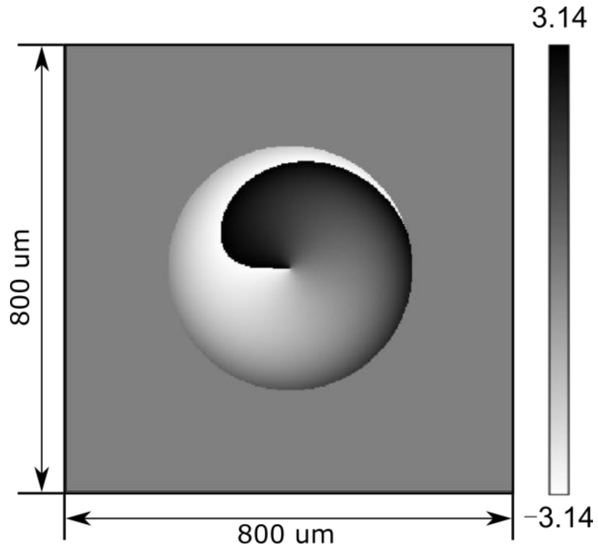
(321 x 321) sampling points

whole phase



(51 x 51) sampling points

residual phase



- When the sampling of whole field is harder than the residual field,

$$N\left[V_\ell\left(\boldsymbol{\rho}\right)\right] > N\left[U_\ell^{\mathrm{res}}\left(\boldsymbol{\rho}\right)\right]$$

we propose to

- store the *residual field information numerically*
- but record *the factor of quadratic phase analytically*

$$\psi_{\mathrm{q}}(\boldsymbol{\rho}) = D_x x^2 + Cxy + D_y y^2$$

Derivation of Semi-analytical Fourier transform (SAFT)

$$V_\ell(\boldsymbol{\rho}) = U_\ell^{\mathrm{res}}(\boldsymbol{\rho}) e^{\mathrm{i}\psi_{\mathrm{q}}(\boldsymbol{\rho})} \xrightarrow{\boldsymbol{\mathcal{F}}} \tilde{V}_\ell(\boldsymbol{\kappa}) = \frac{1}{2\pi} \tilde{U}_\ell^{\mathrm{res}}(\boldsymbol{\kappa}) * \boldsymbol{\mathcal{F}}\left[e^{\mathrm{i}\psi_{\mathrm{q}}(\boldsymbol{\rho})}\right]$$

$$\psi_{\mathrm{q}}(\boldsymbol{\rho}) = D_x x^2 + Cxy + D_y y^2$$

convolution: 2D integral, requires the well sampled field

- Fourier transform of quadratic phase term
  - Mathematic tool: Fresnel Integral

$$\int_{-\infty}^{\infty} e^{-ax^2+bx+c} \mathrm{d}x = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}$$

  - Analytical expression:

$$\boldsymbol{\mathcal{F}}\left[e^{\mathrm{i}\psi_{\mathrm{q}}(\boldsymbol{\rho})}\right] = \sqrt{\frac{\mathrm{i}}{D_x}} \sqrt{\frac{D_x}{\mathrm{i}\gamma}} e^{\mathrm{i}\tilde{\phi}_{\mathrm{q}}(\boldsymbol{\kappa})}$$

with

$$\tilde{\phi}_{\mathrm{q}}(\boldsymbol{\kappa}) = \frac{D_y k_x^2 - C k_x k_y + D_x k_y^2}{\gamma}$$

$$\gamma = C^2 - 4 D_x D_y$$

Derivation of Semi-analytical Fourier transform (SAFT)

- Solving the convolution:

$$\tilde{V}_\ell(\boldsymbol{\kappa}) = \frac{1}{2\pi}\tilde{U}_\ell^{\mathrm{res}}(\boldsymbol{\kappa}) * \boldsymbol{\mathcal{F}}\left[e^{\mathrm{i}\psi_{\mathrm{q}}(\boldsymbol{\rho})}\right]$$

$$= \frac{1}{2\pi}\iint \tilde{U}_\ell^{\mathrm{res}}(\boldsymbol{\kappa}')\sqrt{\frac{\mathrm{i}}{D_x}}\sqrt{\frac{D_x}{\mathrm{i}\gamma}}e^{\mathrm{i}\tilde{\phi}_{\mathrm{q}}(\boldsymbol{\kappa}-\boldsymbol{\kappa}')}\mathrm{d}k_x'\mathrm{d}k_y'$$

$$(\boldsymbol{\kappa}-\boldsymbol{\kappa}')^2 = \boldsymbol{\kappa}^2 - 2\boldsymbol{\kappa}\boldsymbol{\kappa}' + \boldsymbol{\kappa}'^2$$

$$\begin{bmatrix} \beta_x \\ \beta_y \end{bmatrix} = \begin{bmatrix} -\frac{2D_y}{\gamma} & \frac{C}{\gamma} \\ \frac{C}{\gamma} & -\frac{2D_x}{\gamma} \end{bmatrix} \begin{bmatrix} k_{\mathrm{x}}' \\ k_{\mathrm{y}}' \end{bmatrix}$$

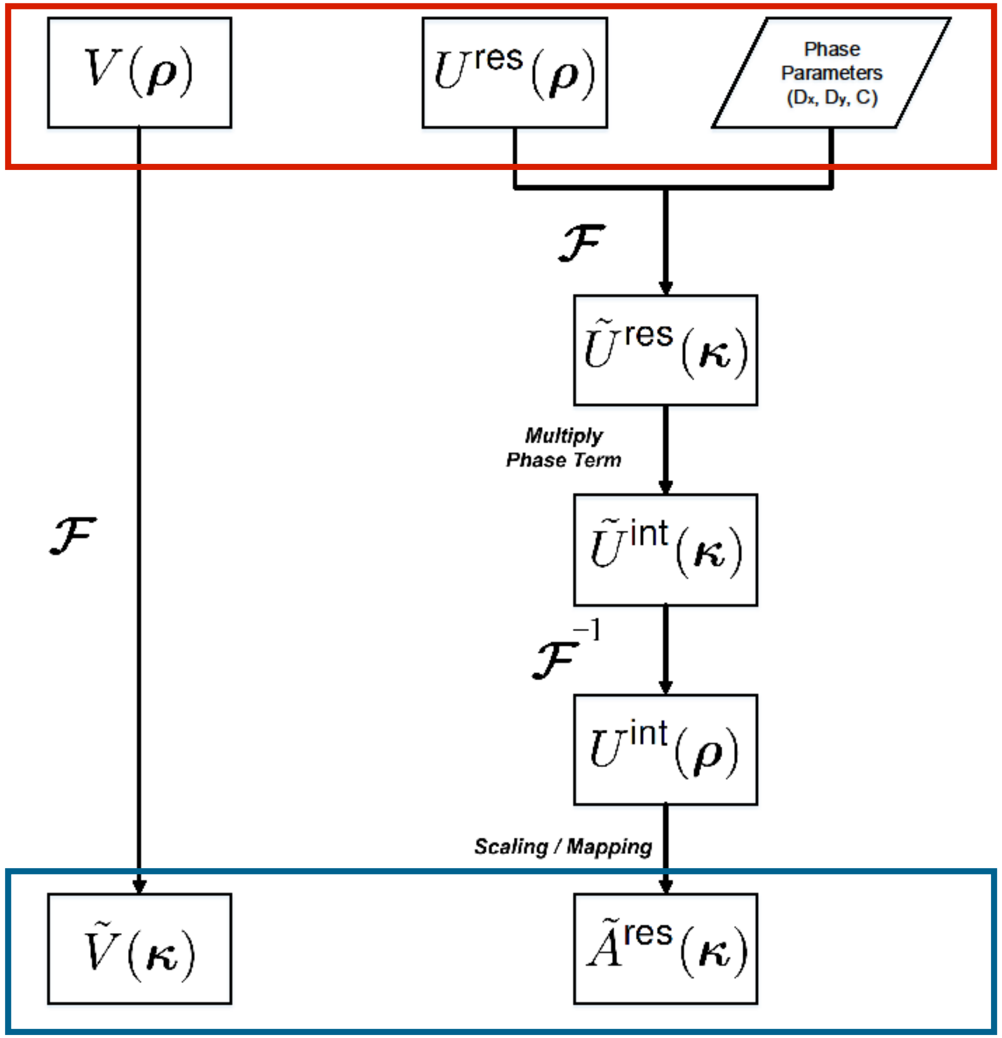
$$\tilde{V}_\ell(\boldsymbol{\kappa}) = \sqrt{\frac{\mathrm{i}}{D_x}}\sqrt{\frac{D_x}{\mathrm{i}\gamma}}e^{\mathrm{i}\tilde{\phi}_{\mathrm{q}}(\boldsymbol{\kappa})}\boldsymbol{\mathcal{F}}_{\boldsymbol{\beta}}^{-1}\left[\tilde{U}_\ell^{\mathrm{res}}(\boldsymbol{\kappa})e^{\mathrm{i}\tilde{\phi}_{\mathrm{q}}(\boldsymbol{\kappa})}\right]$$

$$\boldsymbol{\mathcal{F}}_{\boldsymbol{\beta}}^{-1}\left[\tilde{U}_\ell^{\mathrm{res}}(\boldsymbol{\kappa})e^{\mathrm{i}\tilde{\phi}_{\mathrm{q}}(\boldsymbol{\kappa})}\right] := \tilde{A}_\ell^{\mathrm{res}}(\boldsymbol{\kappa})$$

- Analytical expression of the spectrum

$\tilde{V}_\ell(\boldsymbol{\kappa}) = \tilde{A}_\ell^{\mathrm{res}}(\boldsymbol{\kappa})e^{\mathrm{i}\tilde{\phi}_{\mathrm{q}}(\boldsymbol{\kappa})}$ with $\tilde{\phi}_{\mathrm{q}}(\boldsymbol{\kappa}) = \dfrac{D_y k_x^2 - C k_x k_y + D_x k_y^2}{\gamma}$ and $\gamma = C^2 - 4D_x D_y$

Derivation of Semi-analytical Fourier transform (SAFT)



- Residual field and analytical quadratic phase

$$V_\ell(\boldsymbol{\rho}) = U_\ell^{\text{res}}(\boldsymbol{\rho})e^{\mathrm{i}\psi_{\text{q}}(\boldsymbol{\rho})}$$

$$\psi_{\text{q}}(\boldsymbol{\rho}) = D_x x^2 + Cxy + D_y y^2$$

- 2 x FFT

  2 x pointwise operation

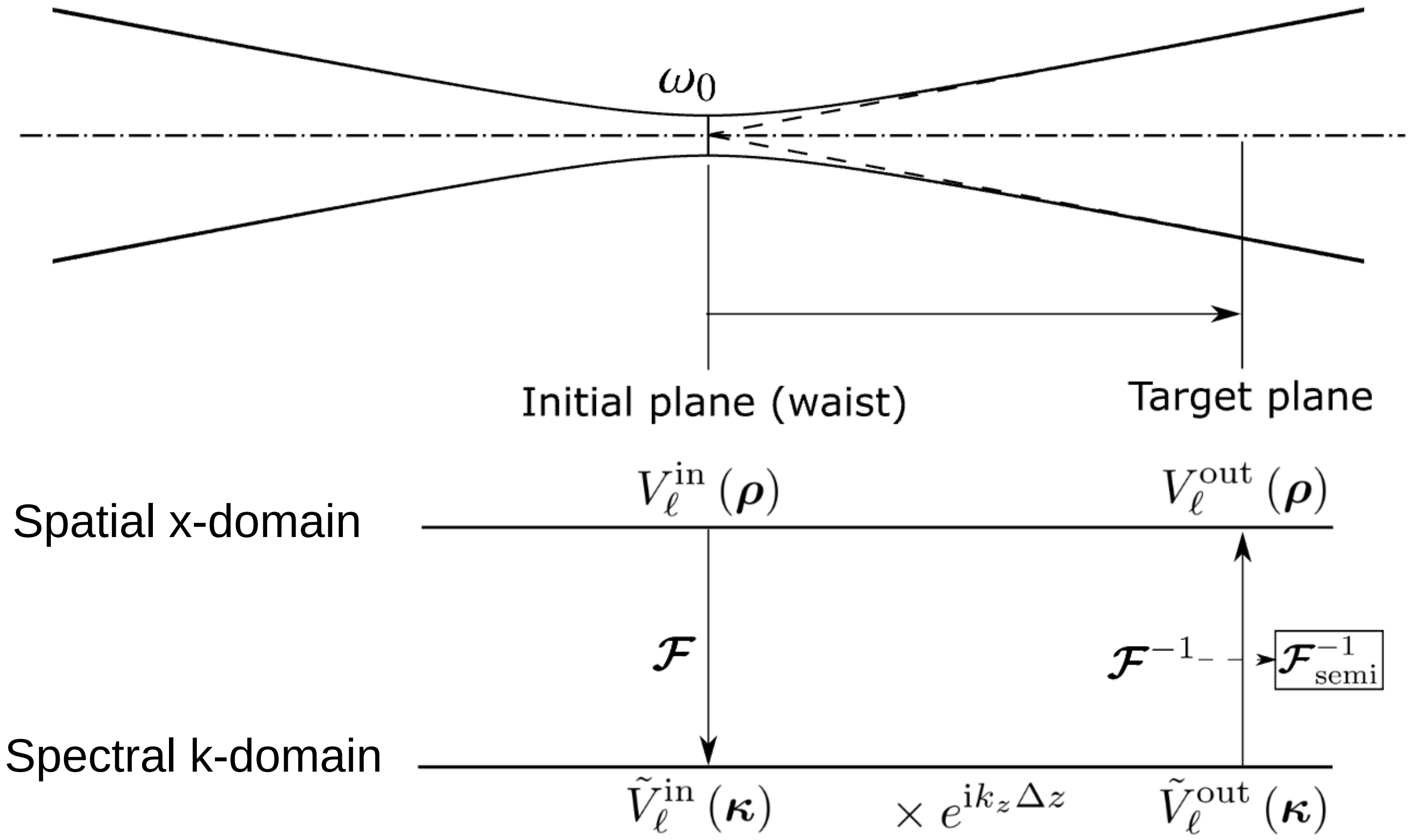
- Residual spectrum and analytical quadratic phase

$$\tilde{V}_\ell(\boldsymbol{\kappa}) = \tilde{A}_\ell^{\text{res}}(\boldsymbol{\kappa})e^{\mathrm{i}\tilde{\phi}_{\text{q}}(\boldsymbol{\kappa})}$$

$$\tilde{\phi}_{\text{q}}(\boldsymbol{\kappa}) = \frac{D_y k_x^2 - C k_x k_y + D_x k_y^2}{\gamma}$$

Application of SAFT to physical-optics modelling

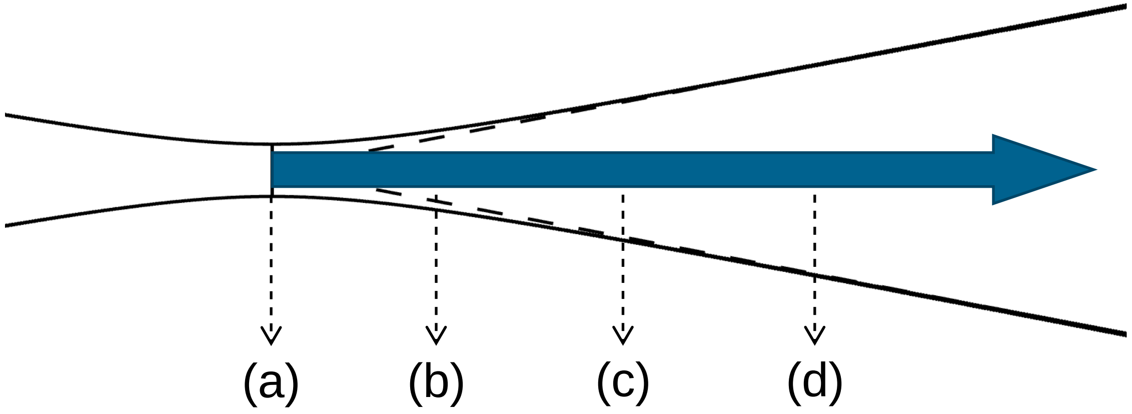
Example 1: Gaussian Beam Propagation



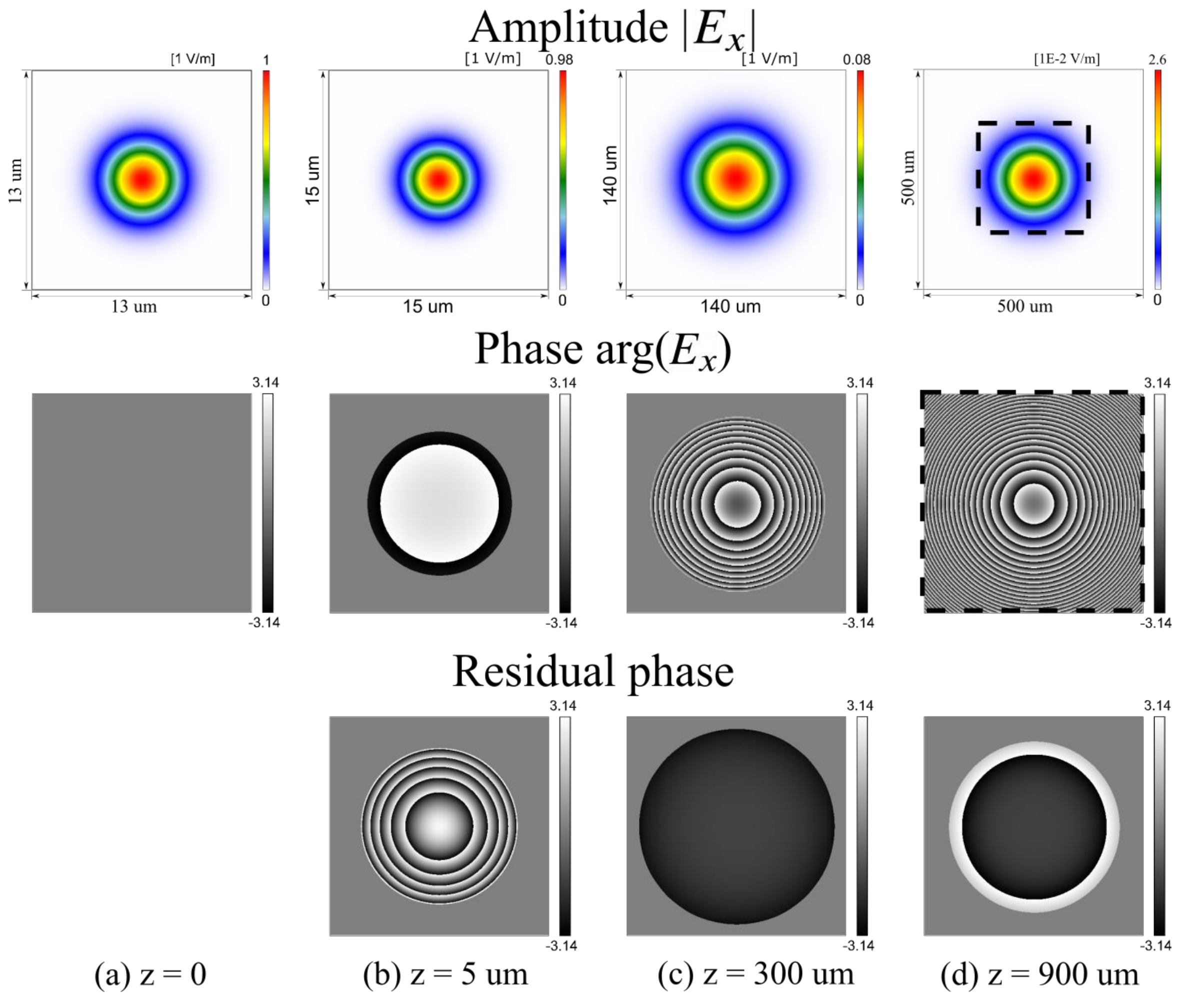
- Rigorous SPW method
  - FFT
  - Propagating kernel
  - IFFT

- Three tests:
  - Paraxial Fundamental Gaussian
  - Non-Paraxial Fundamental Gaussian
  - Laguerre Gaussian

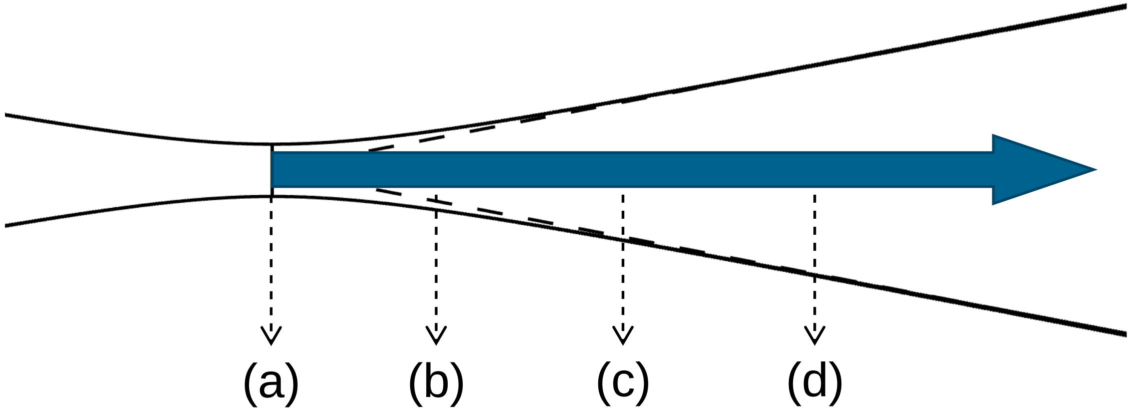
Simulation Results 1.1: Paraxial Fundamental Gaussian



	Distance	Sampling Points
(a)	z = 0 (initial field)	/
(b)	$z = 0.2z_R$ = 5 um	(127 x 127)
(c)	$z = 13z_R$ = 300 um	(31 x 31)
(d)	$z = 38z_R$ = 900 um	(89 x 89)

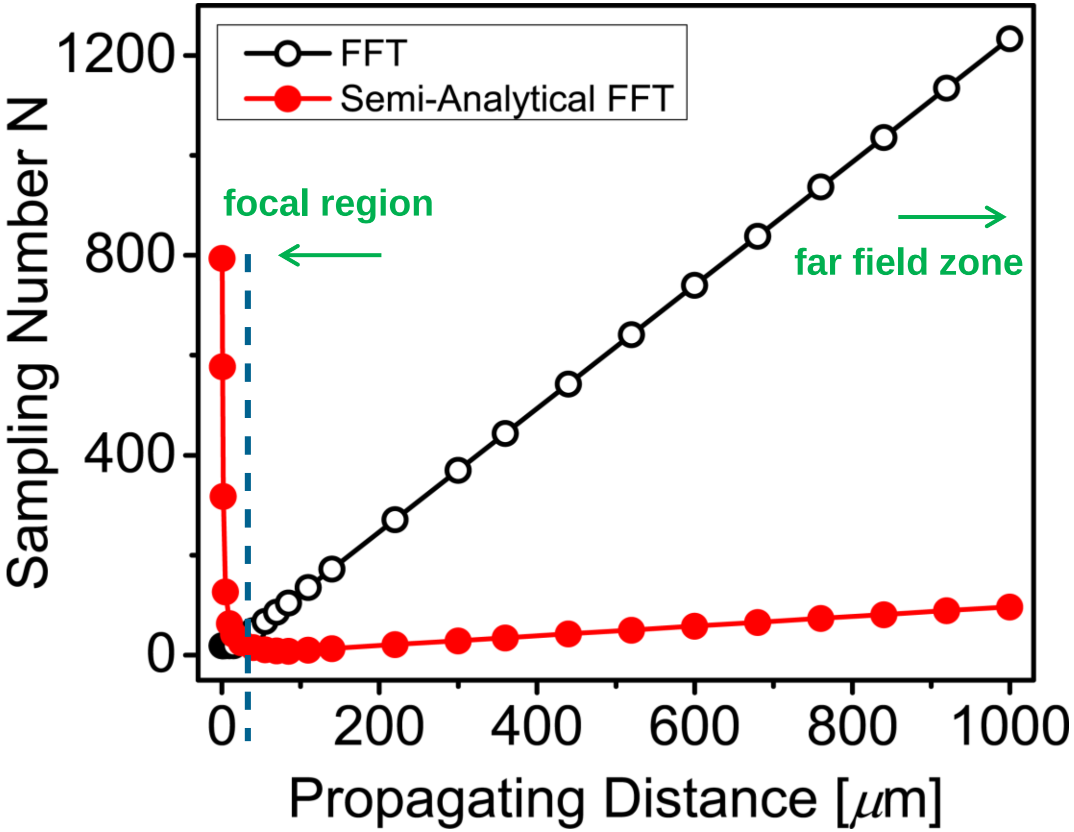


Simulation Results 1.1: Paraxial Fundamental Gaussian

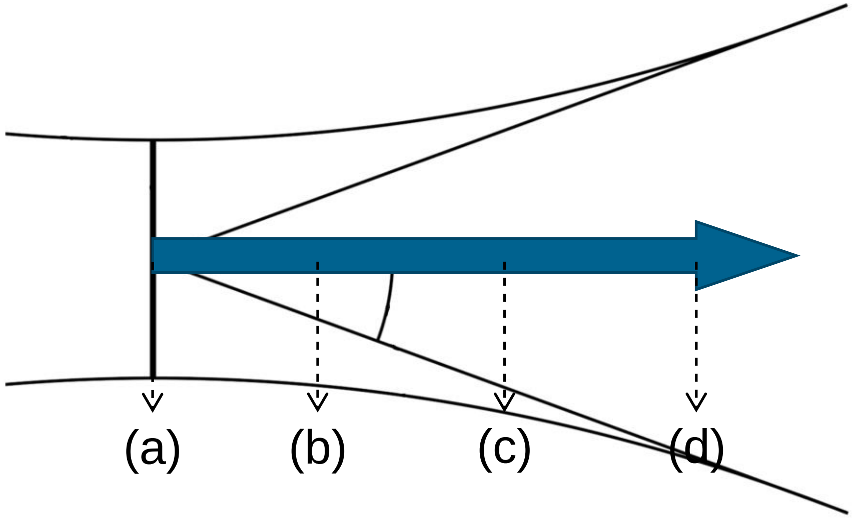


	Distance	Sampling Points
(a)	z = 0 (initial field)	/
(b)	$z = 0.2z_R$ = 5 um	(127 x 127)
(c)	$z = 13z_R$ = 300 um	(31 x 31)
(d)	$z = 38z_R$ = 900 um	(89 x 89)

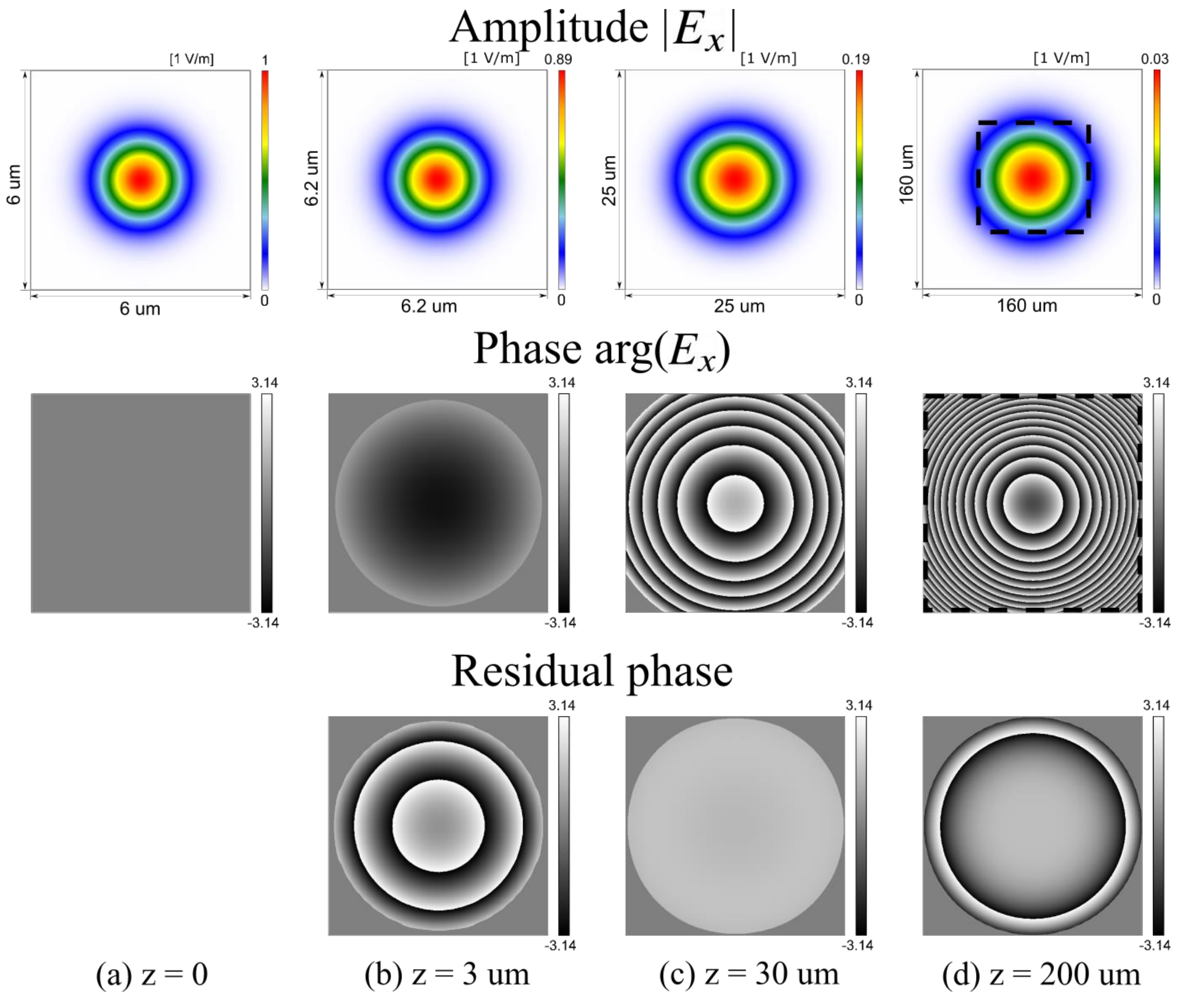
$$e^{ik_z \Delta z}$$



Simulation Results 1.2: Non-Paraxial Fundamental Gaussian

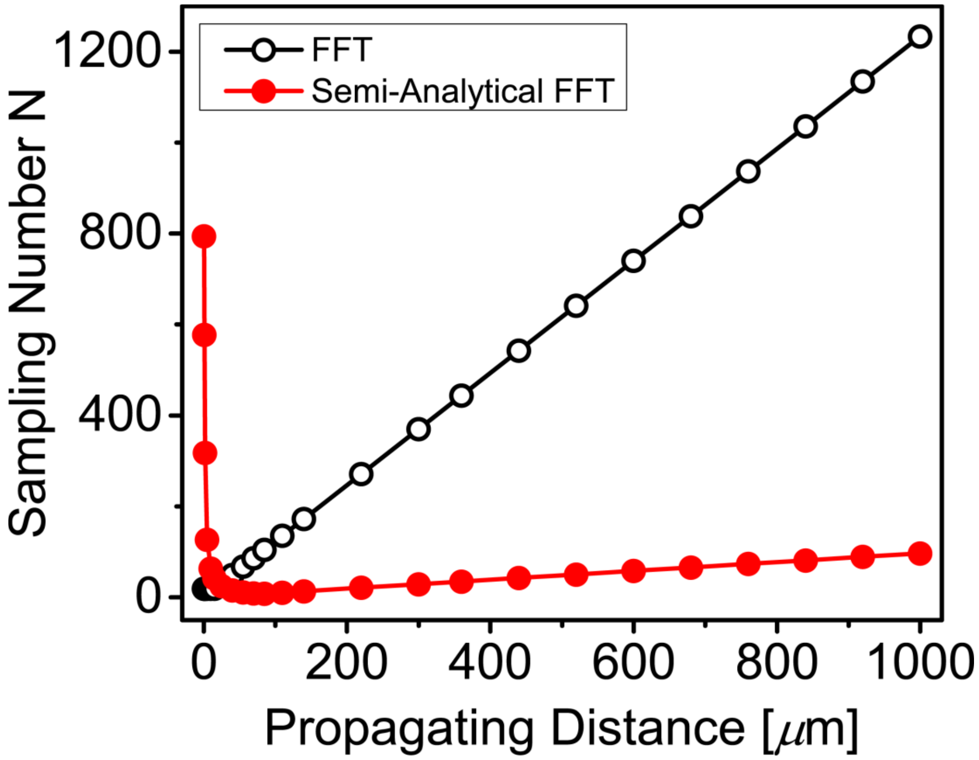


	Distance	Sampling Points
(a)	z = 0 (initial field)	/
(b)	$z = 0.5z_R = 3$ um	(45 x 45)
(c)	$z = 5z_R = 30$ um	(137 x 137)
(d)	$z = 34z_R = 200$ um	(911 x 911)

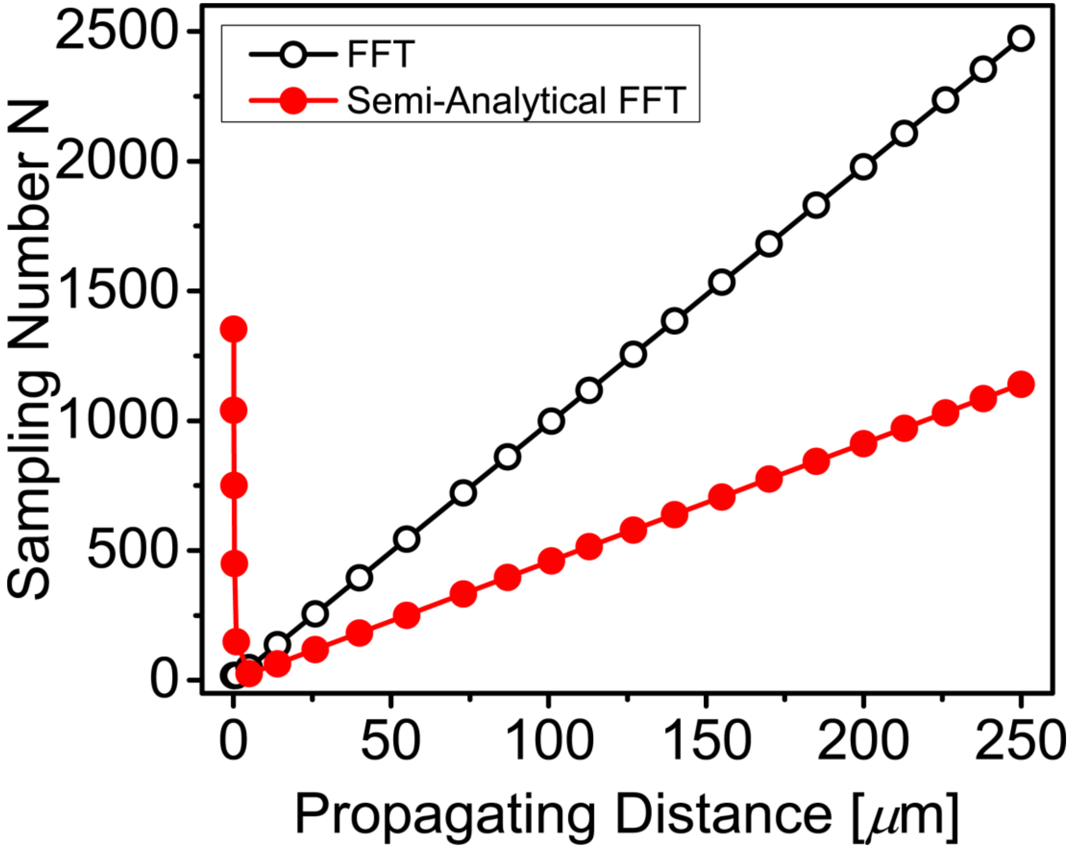


Analysis of Numerical Effort

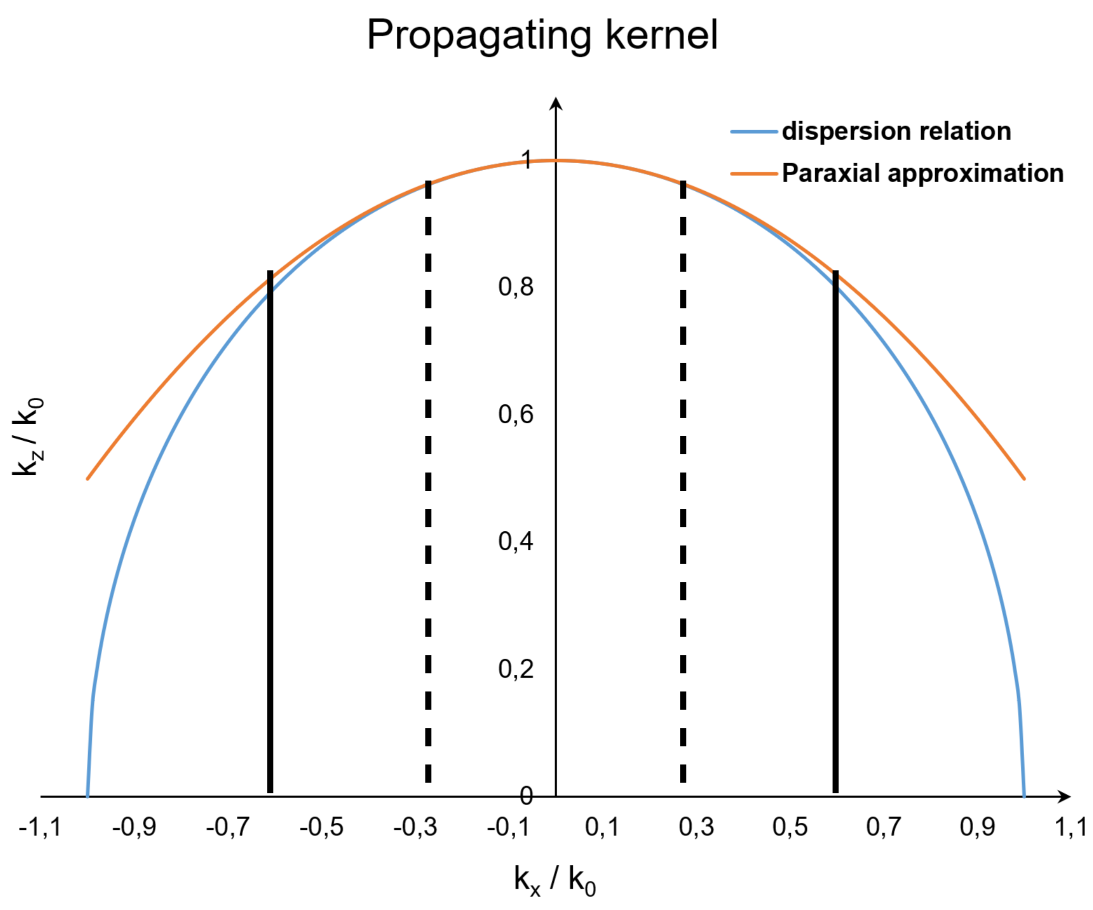
Paraxial Fundamental Gaussian



Non-Paraxial Fundamental Gaussian



Analysis of Numerical Effort



- propagating kernel: $e^{\mathrm{i}k_z(\boldsymbol{\kappa})z}$

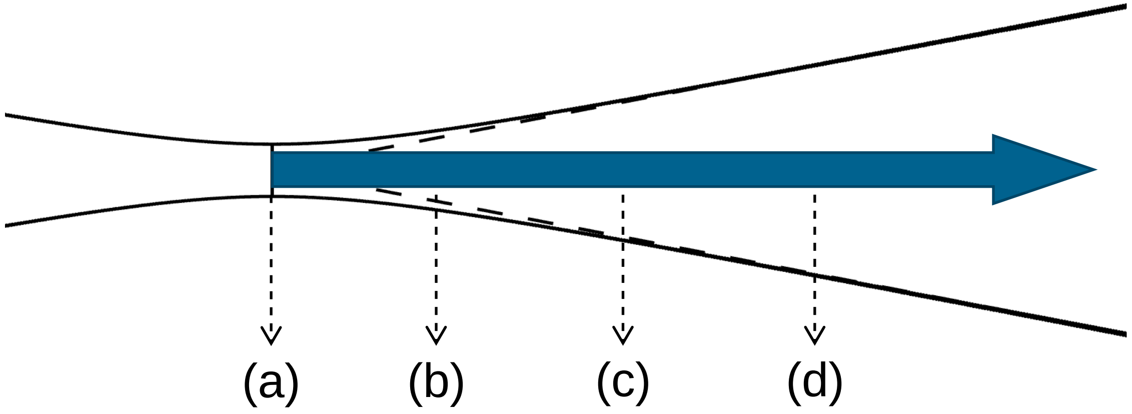
$$k_z(\boldsymbol{\kappa}) = \sqrt{k_0^2 - k_x^2 - k_y^2}$$

- extract the quadratic phase:
  - Taylor expansion around the center: paraxial approximation

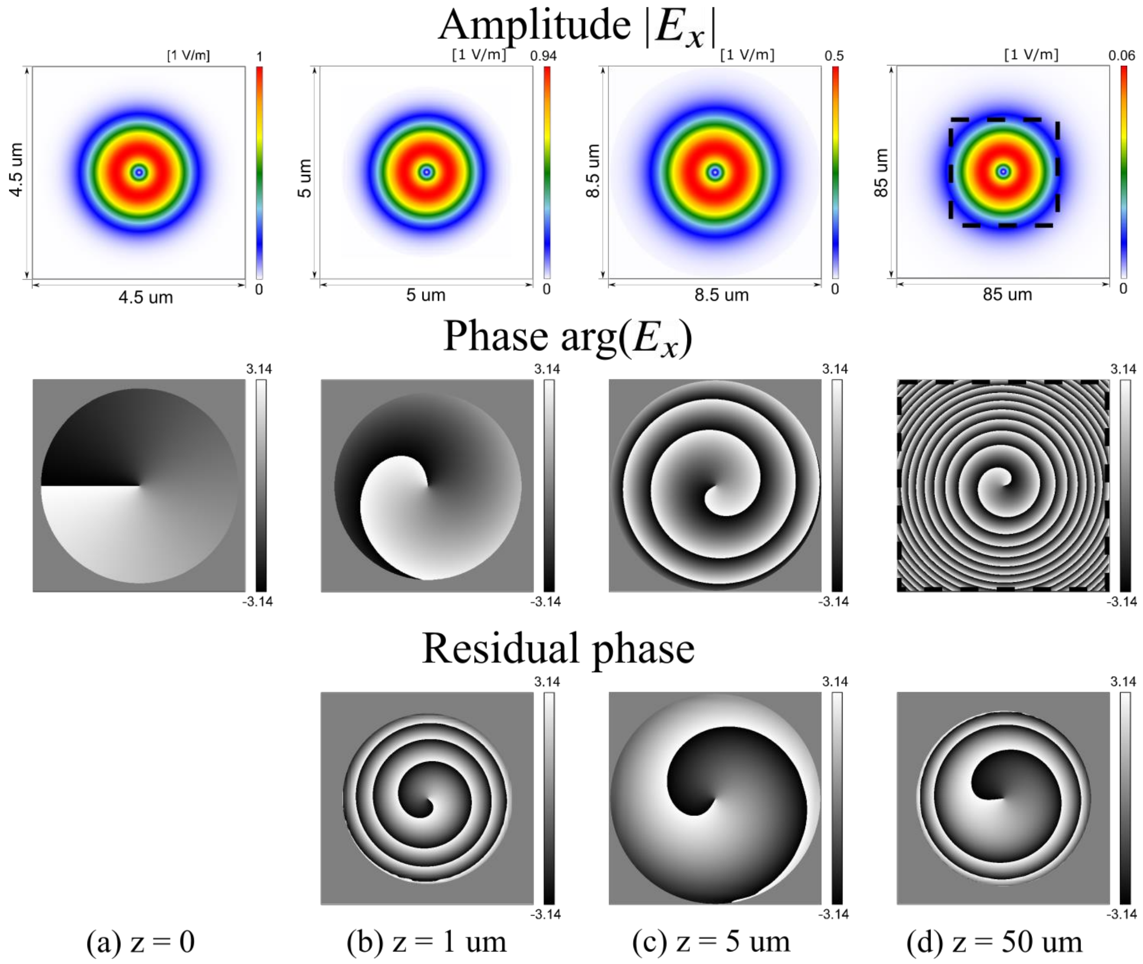
$$k_z(\boldsymbol{\kappa}) \approx k_0 - \frac{1}{2}\left(k_x^2 + k_y^2\right)$$

  - Paraxial beam (dash-line)
  - Non-paraxial beam (solid-line)
- non-paraxial cases: approximation performs worse ➔ more sampling effort of residual field

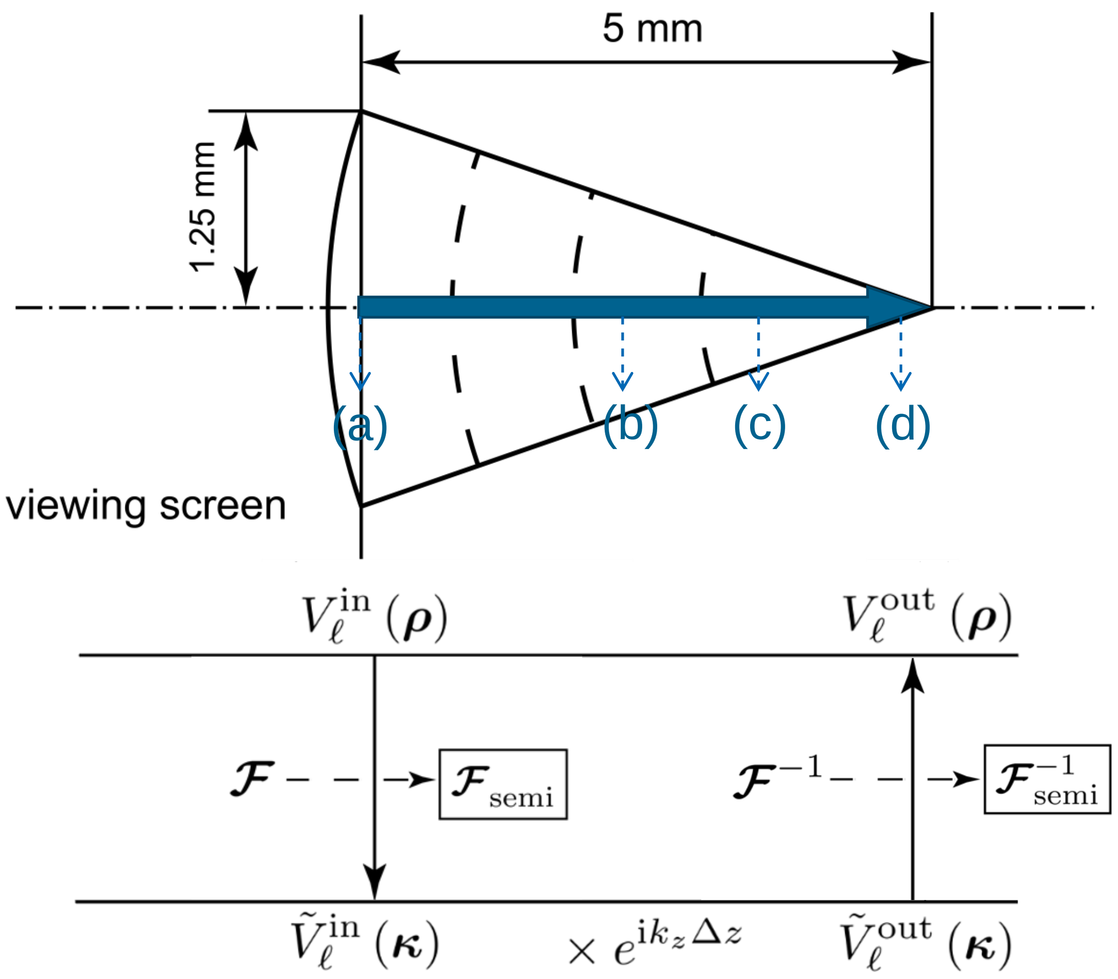
Simulation Results 1.3: Laguerre Gaussian

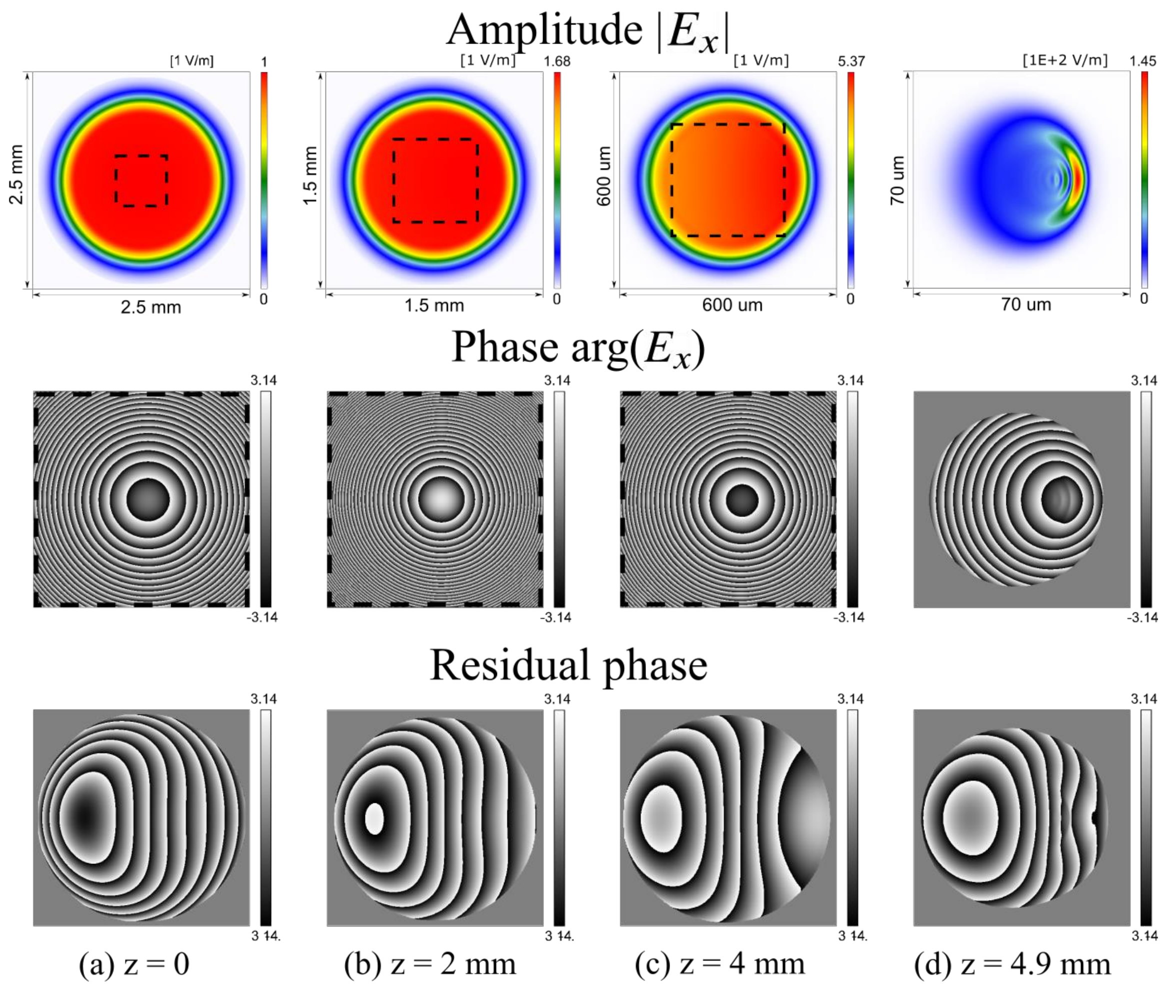


	Distance	Sampling Points
(a)	z = 0 (initial field)	/
(b)	$z = 0.3z_R$ = 1 um	(95 x 95)
(c)	$z = 1.6z_R$ = 5 um	(21 x 21)
(d)	$z = 17z_R$ = 50 um	(157 x 157)

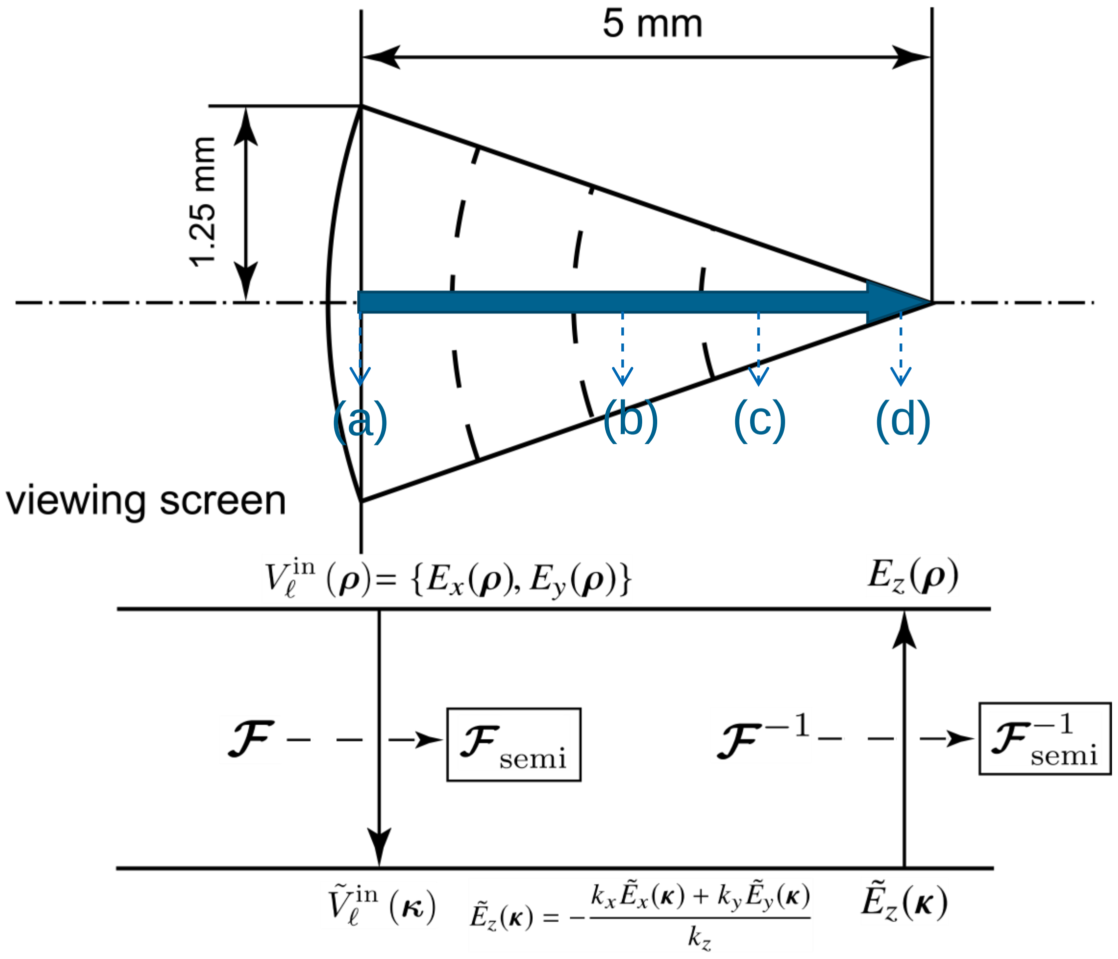


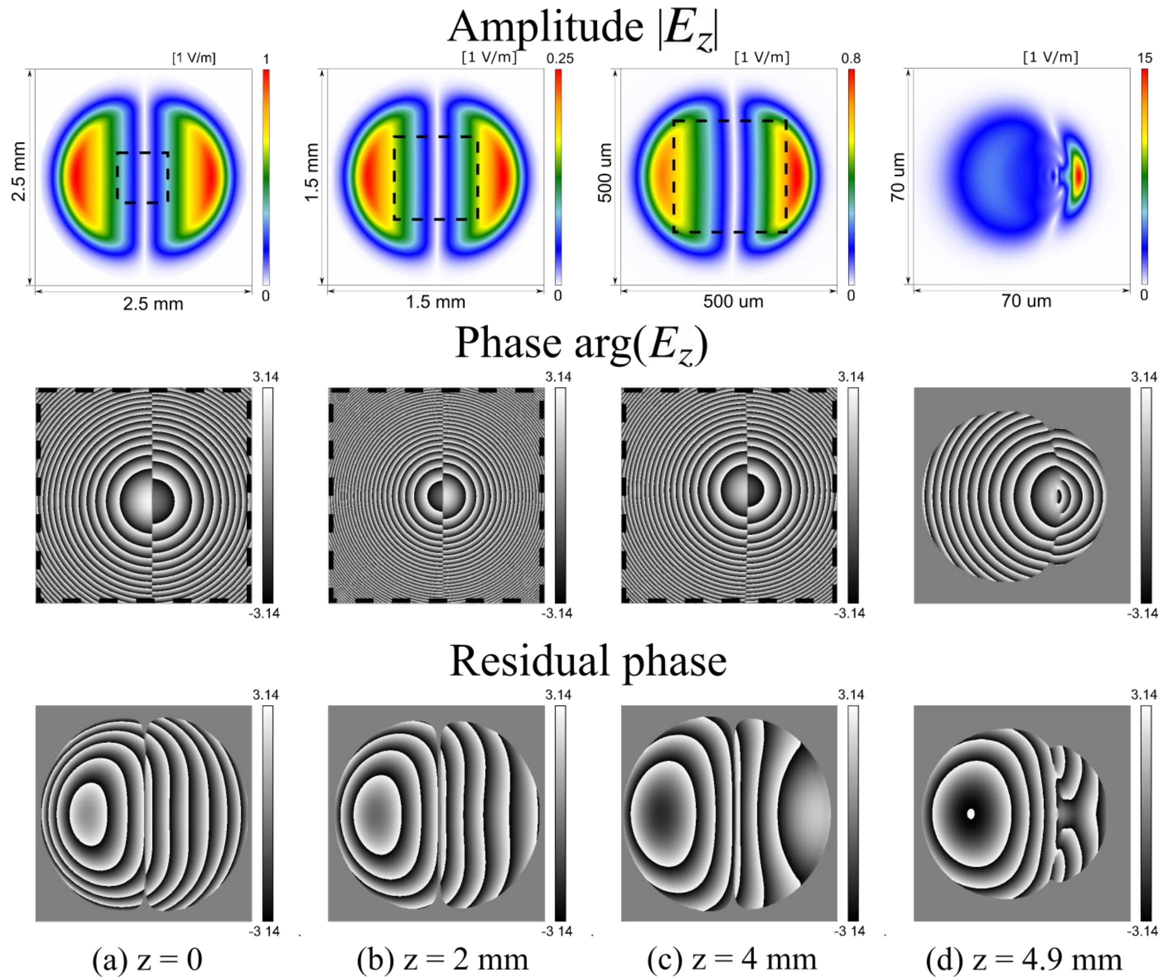
Example 2: (a) Focusing of Aberrated Spherical Wave



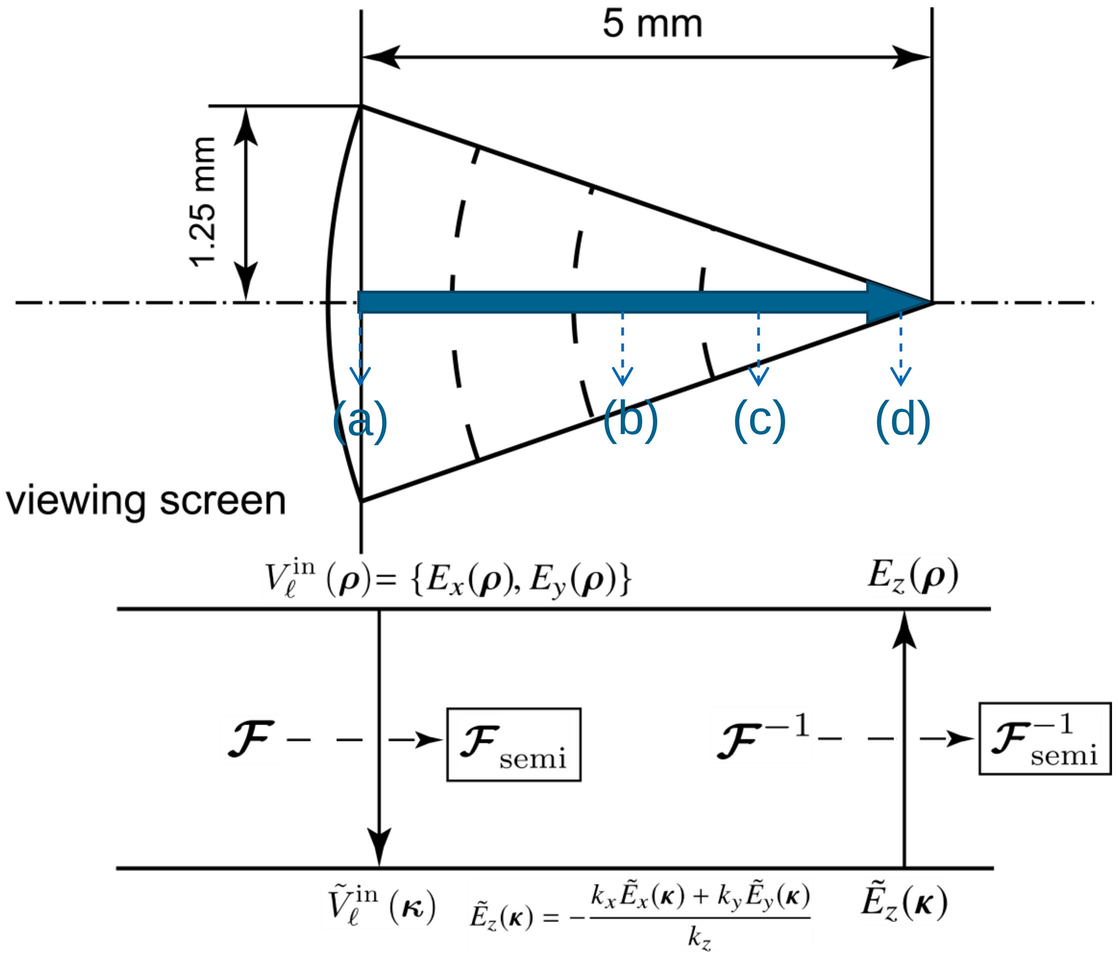


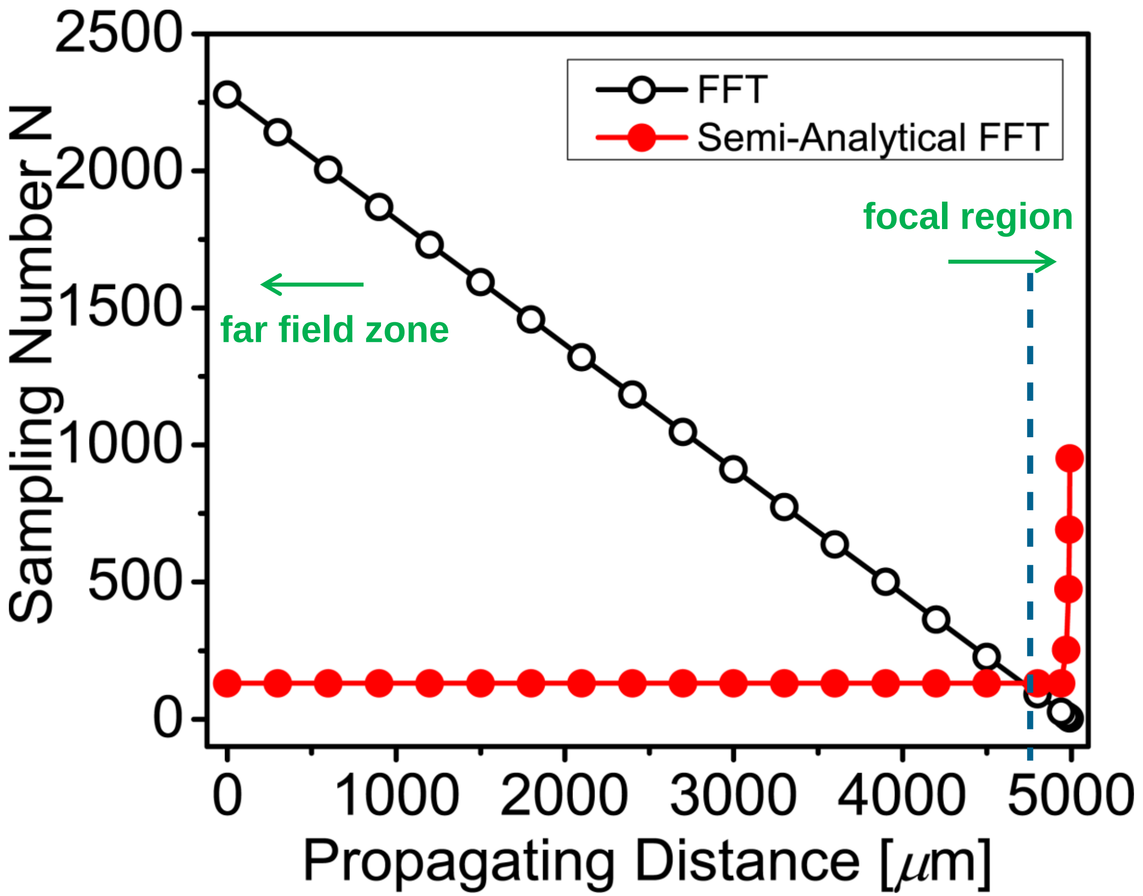
Example 2: (b) E_z Calculation





Example 2: Focusing of Spherical Wave and E_z Calculation





Conclusion

- We present the theory of SAFT
  - analytical handling of quadratic phase
  - rigorous derivation process
  - replace the FFT of the fully sampled field by two FFTs of complex functions which require significantly fewer sampling points

- Two groups of numerical examples are shown to demonstrate the potential of this approach.
  - valid for the field with strong quadratic phase
  - the sampling effort of SAFT only depends on the residual field
  - dramatically reduce the sampling effort

Thank you for your attention!