FRIEDRICH-SCHILLER-UNIVERSITÄT JENA

EOS Topical Meeting on Diffractive Optics 2019

Numerical implementation of the homeomorphic Fourier transform and its application to physical-optics modeling

Zongzhao Wang[1,2], Olga Baladron-Zorita[1,2], Christian Hellmann[2,3] and Frank Wyrowski[1]

[1] Applied Computational Optics Group, Friedrich Schiller University Jena, Germany, 07743

[2] LightTrans International UG, Jena, Germany, 07745

[3] Wyrowski Photonics GmbH, Jena, Germany, 07745

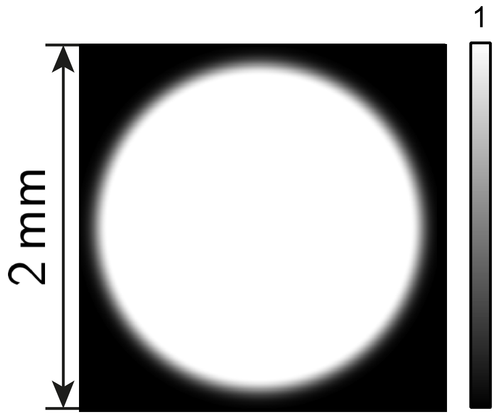
Representation of the Electromagnetic Field

The electromagnetic field at a certain plane is given by

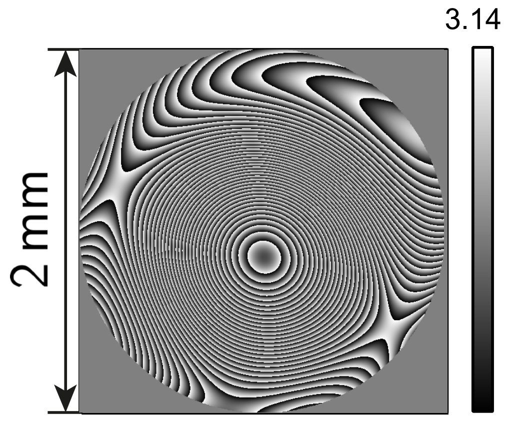
$$V_\ell\left(\boldsymbol{\rho}\right) = \left|V_\ell\left(\boldsymbol{\rho}\right)\right| \exp\left(\mathrm{i}\gamma_\ell(\boldsymbol{\rho})\right)$$

where $\ell = 1, \ldots, 6$ to account for all six field components. $\boldsymbol{\rho} = (x, y)$ is the projection of the position vector onto the transversal plane.

Amplitude: $\left|V_\ell\left(\boldsymbol{\rho}\right)\right|$



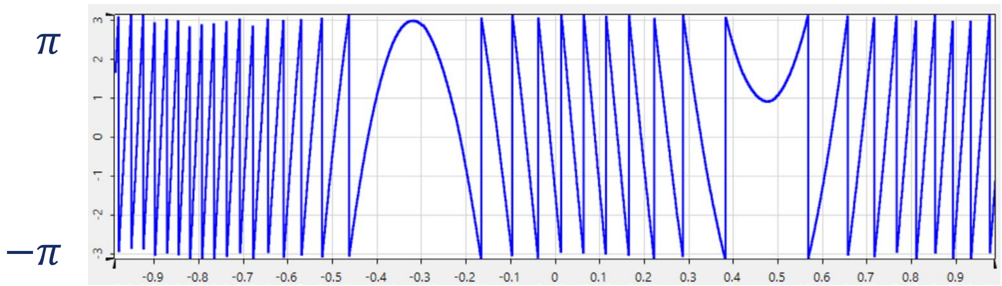
Phase: $\gamma_\ell(\boldsymbol{\rho}) = \arg\left[V_\ell\left(\boldsymbol{\rho}\right)\right]$



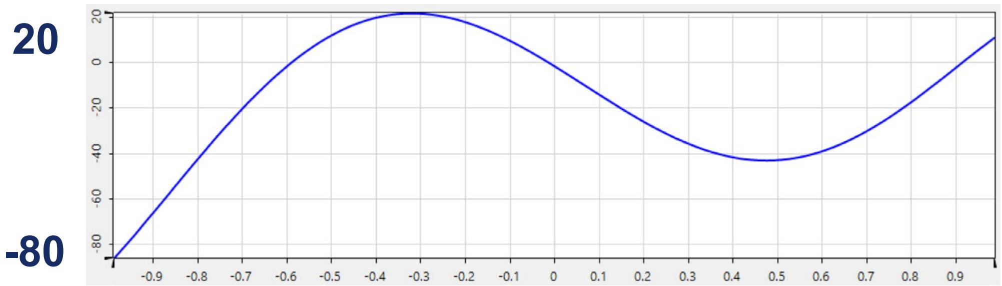
Extract the Smooth Wavefront Phase (Geometric Phase)

$$
\begin{aligned}
V_\ell(\boldsymbol{\rho}) &= |V_\ell(\boldsymbol{\rho})| \exp(\mathrm{i}\gamma_\ell(\boldsymbol{\rho})) \\
&= U_\ell(\boldsymbol{\rho}) \exp(\mathrm{i}\psi(\boldsymbol{\rho}))
\end{aligned}
$$

- Residual diffractive field: $U_\ell(\boldsymbol{\rho}) = |V_\ell(\boldsymbol{\rho})| \exp(\mathrm{i}\gamma_\ell(\boldsymbol{\rho}) - \mathrm{i}\psi(\boldsymbol{\rho}))$
- Smooth wavefront phase: $\psi(\boldsymbol{\rho})$



wrapped $\psi(\boldsymbol{\rho})$ in 2π-modulo



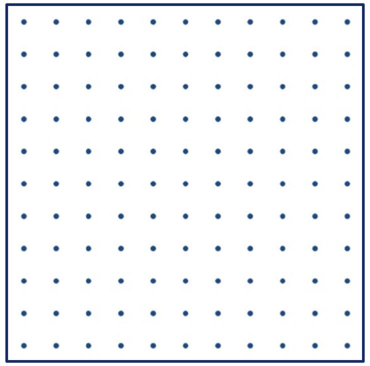
unwrapping $\psi(\boldsymbol{\rho})$

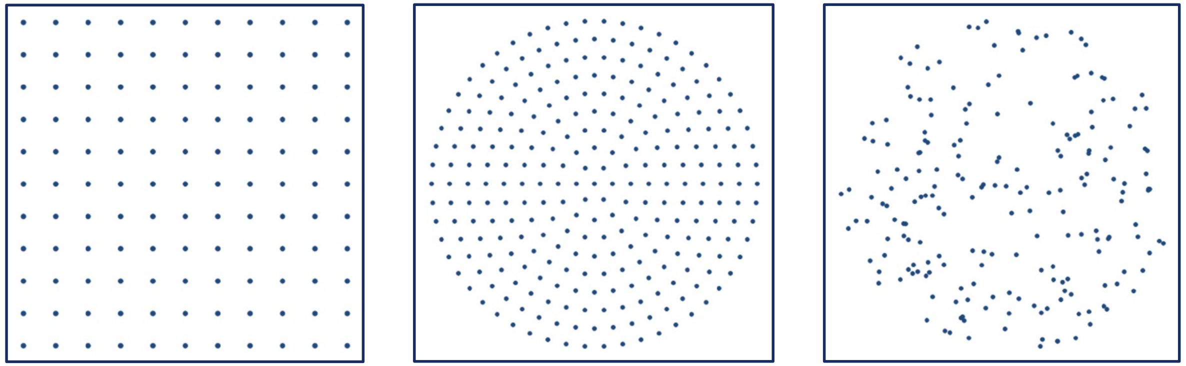
Hybrid Sampling Strategy

$$V_\ell\left(\boldsymbol{\rho}\right) = U_\ell\left(\boldsymbol{\rho}\right) \exp\left(\mathrm{i}\psi(\boldsymbol{\rho})\right)$$

gridded sampling (equidistance)

gridless sampling (non-equidistance)

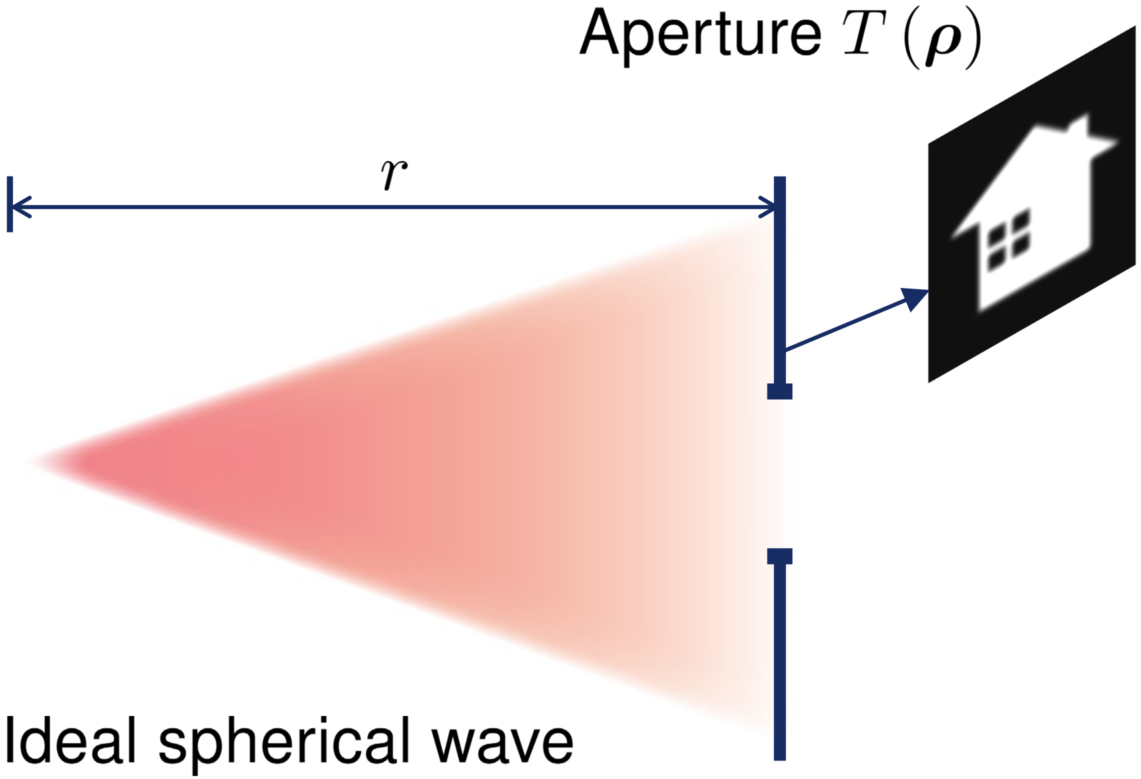




Interpolation methods
- **Sinc interpolation**
- **Cubic 8 interpolation**

Interpolation methods
- **Quadratic interpolation**
- **B-Spline interpolation**

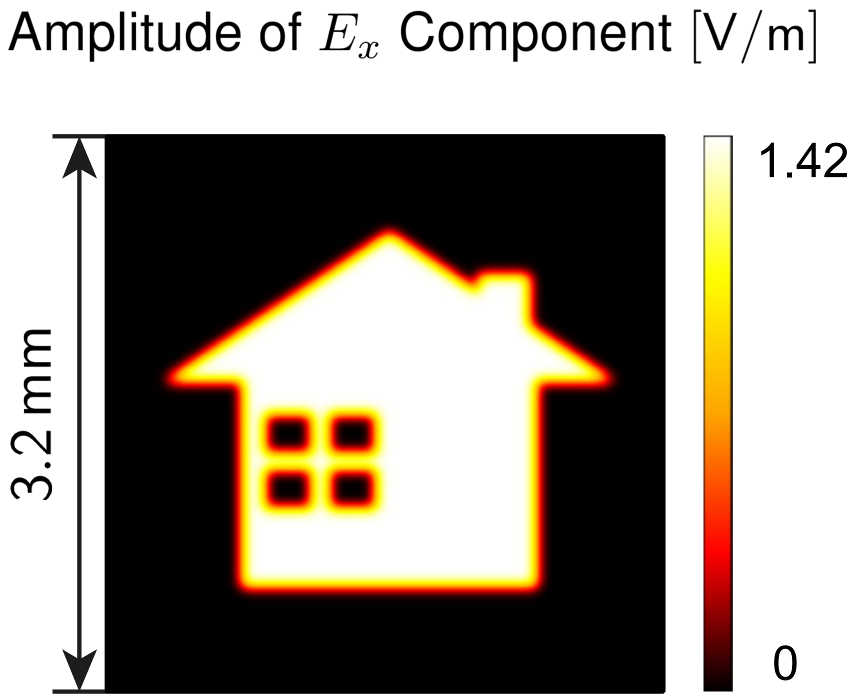
Field behind a Mask



Ideal spherical wave

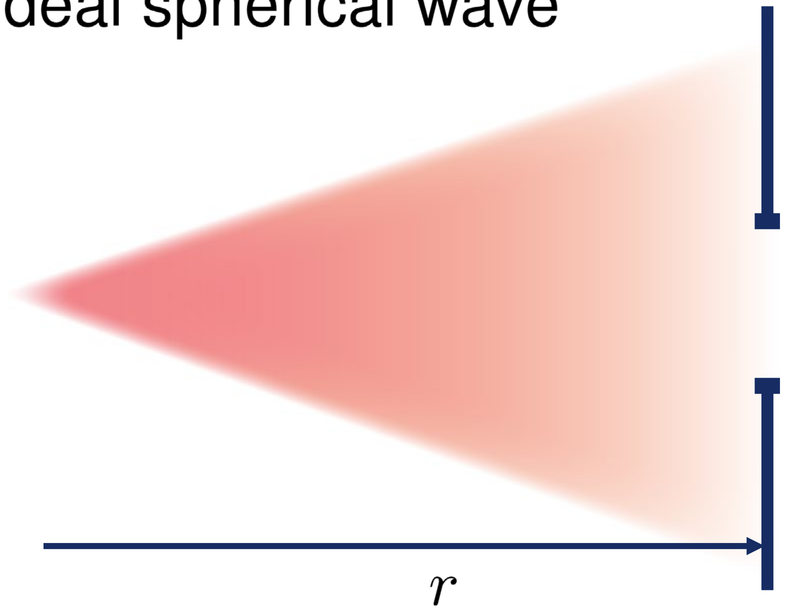
$$V_\ell(\boldsymbol{\rho}) = \frac{1}{\sqrt{\boldsymbol{\rho}^2 + r^2}} \exp\left(\mathrm{i}\psi^{\mathsf{sph}}(\boldsymbol{\rho})\right)$$

$$\psi^{\mathsf{sph}}(\boldsymbol{\rho}) = \operatorname{sgn}(r)\, k_0 \check{n} \sqrt{\boldsymbol{\rho}^2 + r^2}$$



Fourier Transform of the Field behind a Mask

Ideal spherical wave



Aperture $T\left(\boldsymbol{\rho}\right)$

$$V_\ell^{\mathrm{out}}\left(\boldsymbol{\rho}\right) = \frac{T\left(\boldsymbol{\rho}\right)}{\sqrt{\boldsymbol{\rho}^2 + r^2}} \exp\left(\mathrm{i}\psi^{\mathrm{sph}}\left(\boldsymbol{\rho}\right)\right)$$

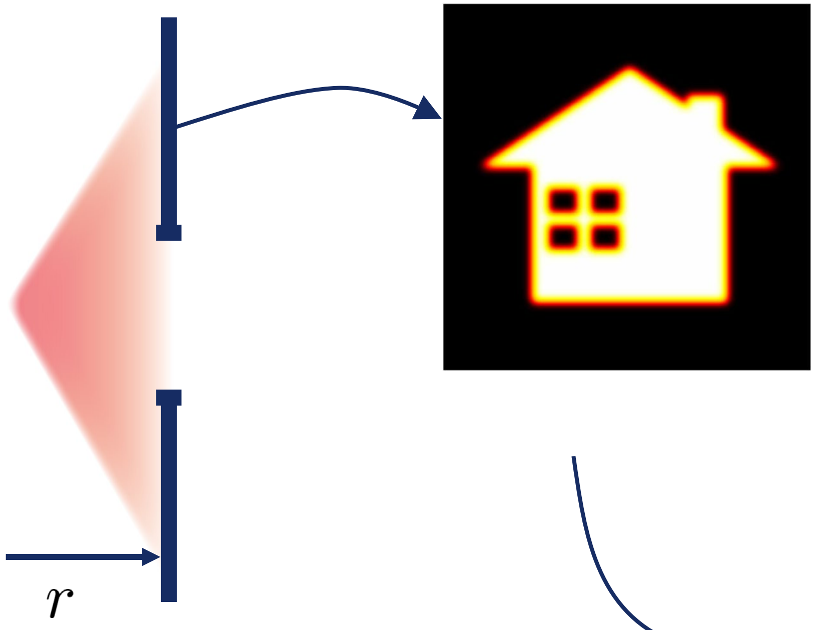
$\boldsymbol{\mathcal{F}}_{\boldsymbol{\kappa}}$

?

$\tilde{V}_\ell^{\mathrm{out}}\left(\boldsymbol{\kappa}\right)$

Fourier Transform of the Field behind the Mask

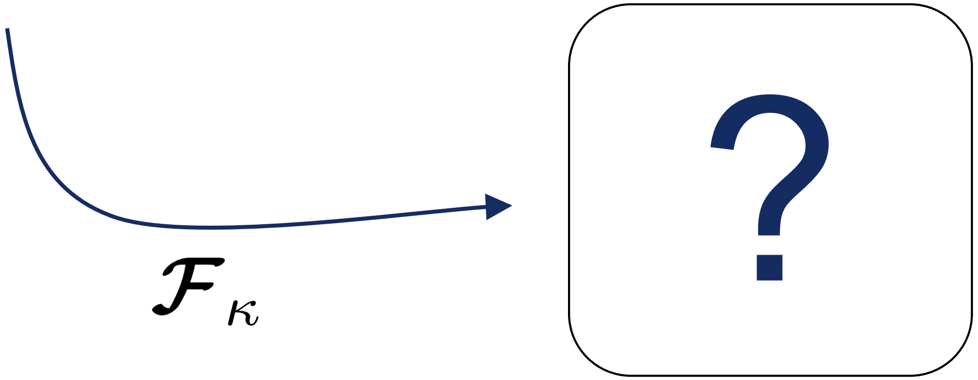
Ideal spherical wave



$$V_\ell^{\mathsf{out}}(\boldsymbol{\rho}) = \frac{T(\boldsymbol{\rho})}{\sqrt{\boldsymbol{\rho}^2 + r^2}} \exp\left(\mathrm{i}\psi^{\mathsf{sph}}(\boldsymbol{\rho})\right)$$

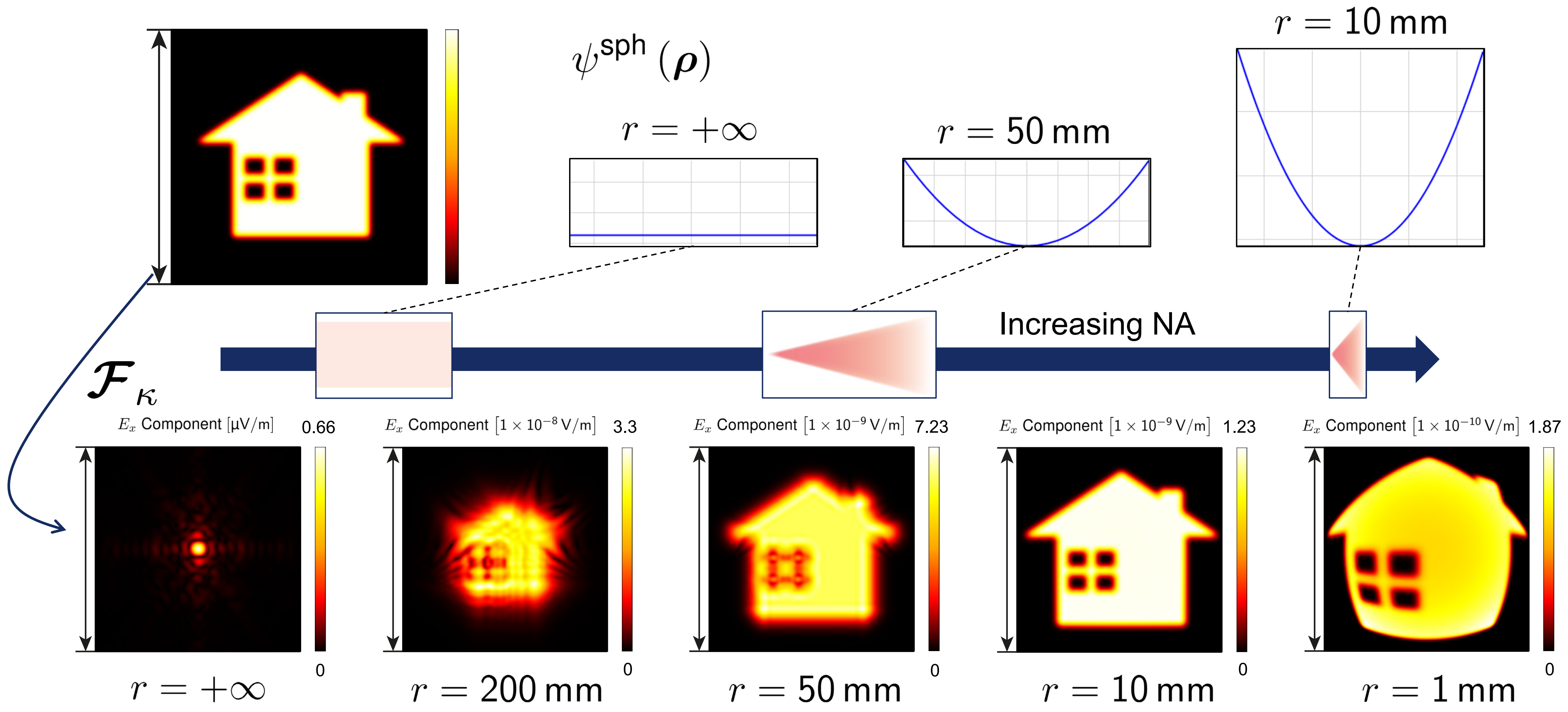
Aperture $T(\boldsymbol{\rho})$

$\mathcal{F}_{\boldsymbol{\kappa}}$

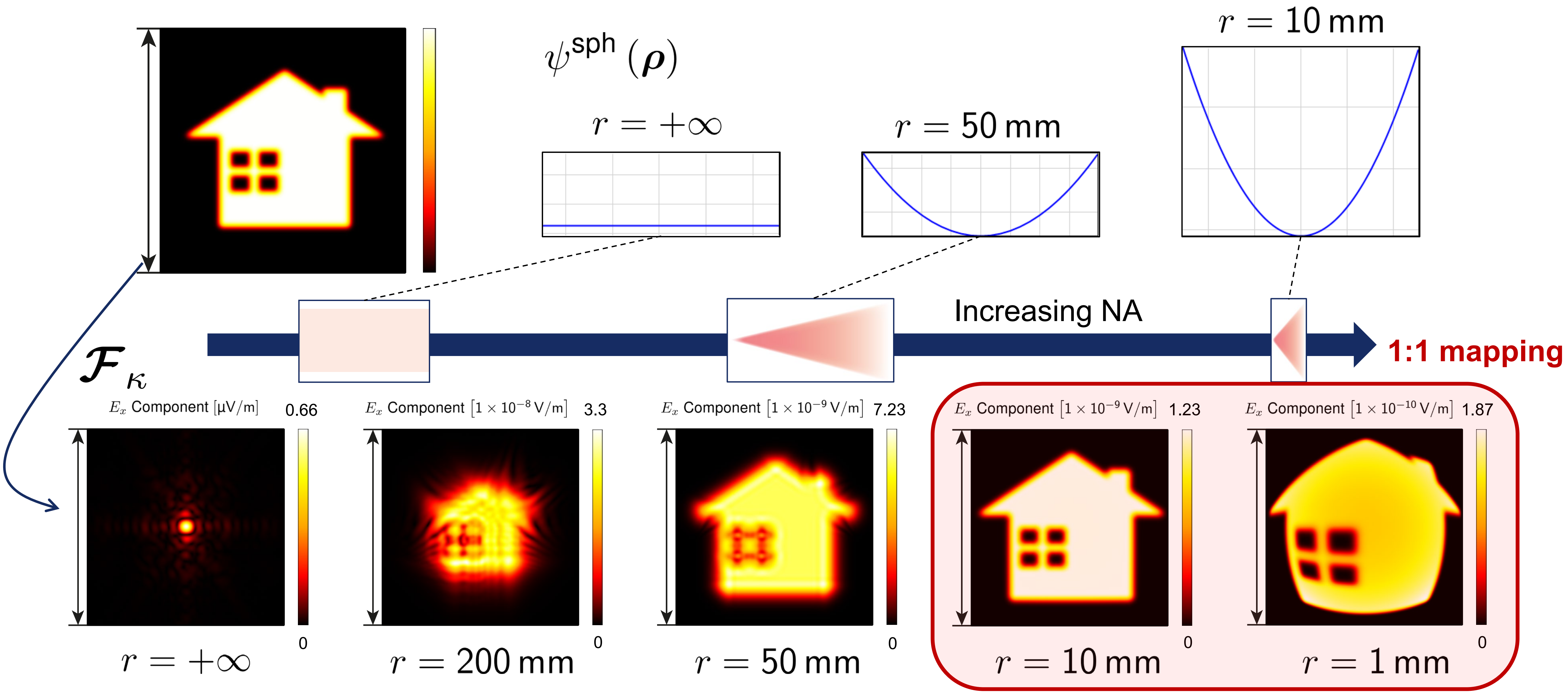


$\tilde{V}_\ell^{\mathsf{out}}(\boldsymbol{\kappa})$

Results of Fourier Transform



Results of Fourier Transform



Homeomorphic Fourier Transform (HFT)

$$V_\ell(\boldsymbol{\rho}) = U_\ell(\boldsymbol{\rho}) \exp(\mathrm{i}\psi(\boldsymbol{\rho}))$$

2D Fourier transform integral

$$\tilde{V}_\ell(\boldsymbol{\kappa}) = \iint_{-\infty}^{\infty} V_\ell(\boldsymbol{\rho}) \exp(-\mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho})\, \mathrm{d}k_x \mathrm{d}k_y$$

$$\boldsymbol{\kappa} = (k_x, k_y)$$

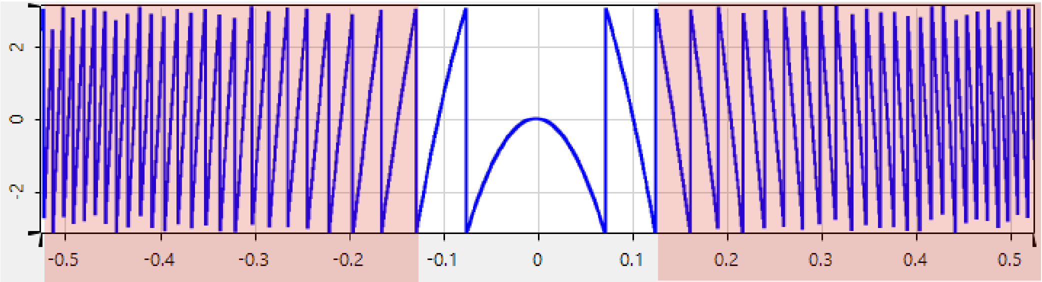
$$\tilde{V}_\ell(\boldsymbol{\kappa}) = \iint_{-\infty}^{\infty} U_\ell(\boldsymbol{\rho}) \exp(\mathrm{i}\psi(\boldsymbol{\rho}) - \mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho})\, \mathrm{d}k_x \mathrm{d}k_y$$

Homeomorphic Fourier Transform (HFT)

$$\tilde{V}_\ell(\boldsymbol{\kappa}) = \iint_{-\infty}^{\infty} U_\ell(\boldsymbol{\rho}) \exp\left(\mathrm{i}\psi(\boldsymbol{\rho}) - \mathrm{i}\boldsymbol{\kappa}\cdot\boldsymbol{\rho}\right) \mathrm{d}k_x \mathrm{d}k_y$$

Stationary phase method

$\psi(\boldsymbol{\rho}) - \boldsymbol{\kappa}\cdot\boldsymbol{\rho}$



at $\boldsymbol{\kappa}$ = (0, 0)

=0

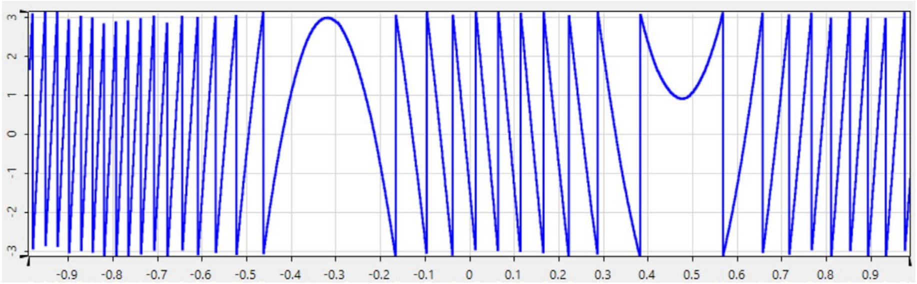
=0

$\boldsymbol{\rho}$ = (0, 0)

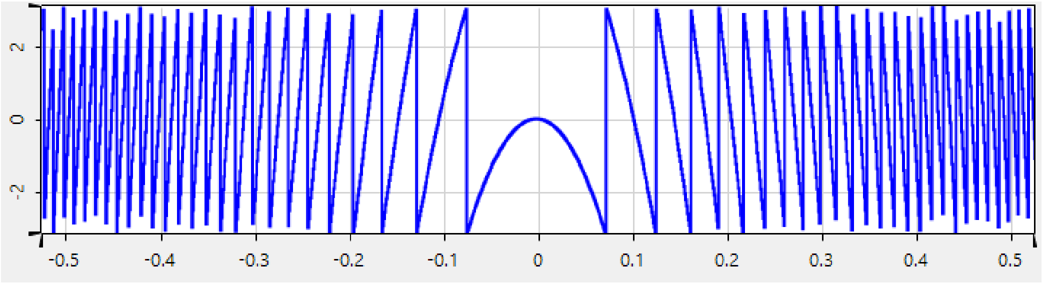
1:1 mapping

$$\boldsymbol{\nabla}\left(\psi(\boldsymbol{\rho}) - \boldsymbol{\kappa}\cdot\boldsymbol{\rho}\right) = 0$$

Bijective Mapping (Homeomorphism)



2:1 mapping (✗)



1:1 mapping (✓)

Conditions of the stationary phase method

- $U_\ell\left(\boldsymbol{\rho}\right)$ is slow varying field. (wavefront phase dominate FT)
- For given $\boldsymbol{\kappa}$, there is only one $\boldsymbol{\rho}$ have $\boldsymbol{\nabla}\left(\psi(\boldsymbol{\rho})-\boldsymbol{\kappa}\cdot\boldsymbol{\rho}\right)=0$. (bijective mapping)

Criteria of the homeomorphism: the second derivative factor $\psi_{xy}^2-\psi_{xx}\psi_{yy}$ are all positive or all negative in the definition domain.

Homeomorphic Fourier transform

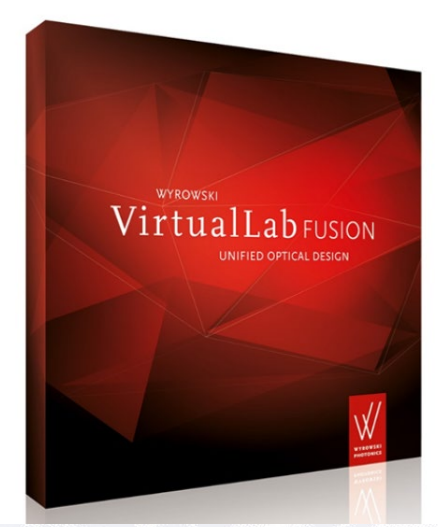
$$V_\ell(\boldsymbol{\rho}) = U_\ell(\boldsymbol{\rho}) \exp(\mathrm{i}\psi(\boldsymbol{\rho}))$$

$$\nabla\psi(\boldsymbol{\rho}) = \boldsymbol{\kappa}$$ ($\boldsymbol{\rho} \to \boldsymbol{\kappa}$, 1:1 mapping relation)

$$\begin{aligned}\tilde{V}_\perp(\boldsymbol{\kappa}) &= \alpha[\boldsymbol{\rho}(\boldsymbol{\kappa})]\, U_\ell[\boldsymbol{\rho}(\boldsymbol{\kappa})] \exp(\mathrm{i}\psi[\boldsymbol{\rho}(\boldsymbol{\kappa})] - \boldsymbol{\kappa}\cdot\boldsymbol{\rho}(\boldsymbol{\kappa})) \\ &= \tilde{A}(\boldsymbol{\kappa}) \exp\left(\mathrm{i}\tilde{\psi}(\boldsymbol{\kappa})\right)\end{aligned}$$

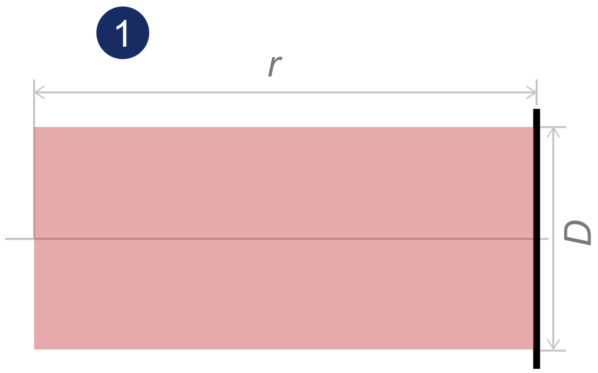
with $$\alpha(\boldsymbol{\rho}) = \begin{cases} \sqrt{\frac{\mathrm{i}}{\psi_{xx}(\boldsymbol{\rho})}}\sqrt{-\frac{\mathrm{i}\psi_{xx}(\boldsymbol{\rho})}{\psi_{xy}^2(\boldsymbol{\rho})-\psi_{xx}(\boldsymbol{\rho})\psi_{yy}(\boldsymbol{\rho})}} & , \quad \psi_{xx}(\boldsymbol{\rho}) \neq 0 \\ \frac{1}{|\psi_{xy}(\boldsymbol{\rho})|} & , \quad \psi_{xx}(\boldsymbol{\rho}) = 0 \end{cases}$$

Numerical implementation and examples



HFT vs. FFT @ NA = 0

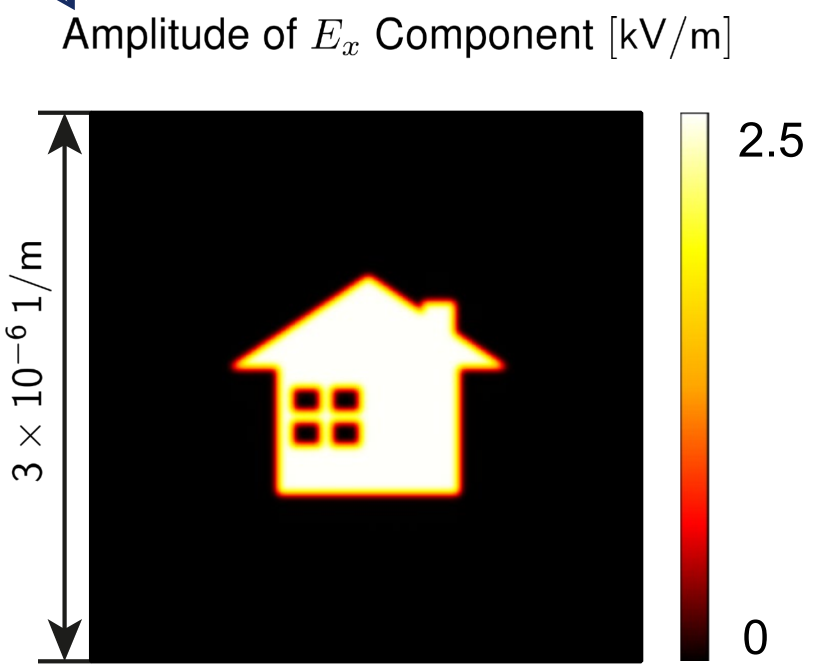
Sampling Points for Different Fourier Transform Techniques		
	HFT	FFT
NA = 0	(73 x 59), not accurate	(73 x 59)



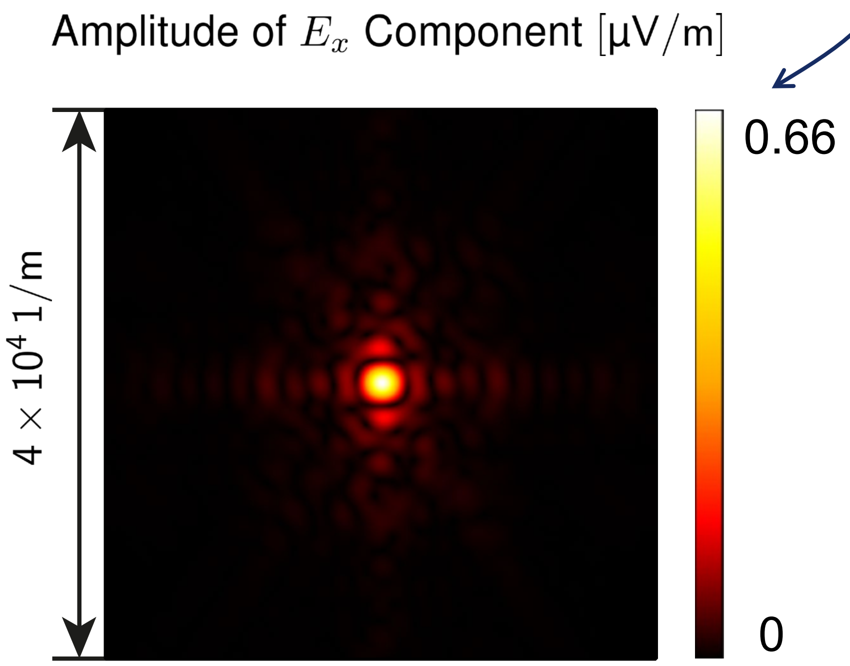
r = inf,

D = 2.7 mm

➔ NA = 0



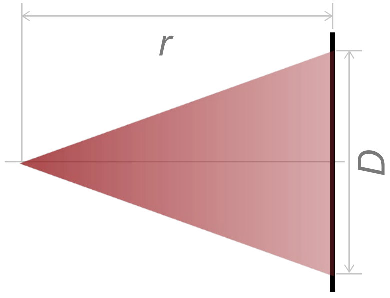
quite small window size



HFT vs. FFT @ NA = 0.027

Sampling Points for Different Fourier Transform Techniques		
	HFT	FFT
NA = 0.027	(73 x 59)	(7947 x 6225)

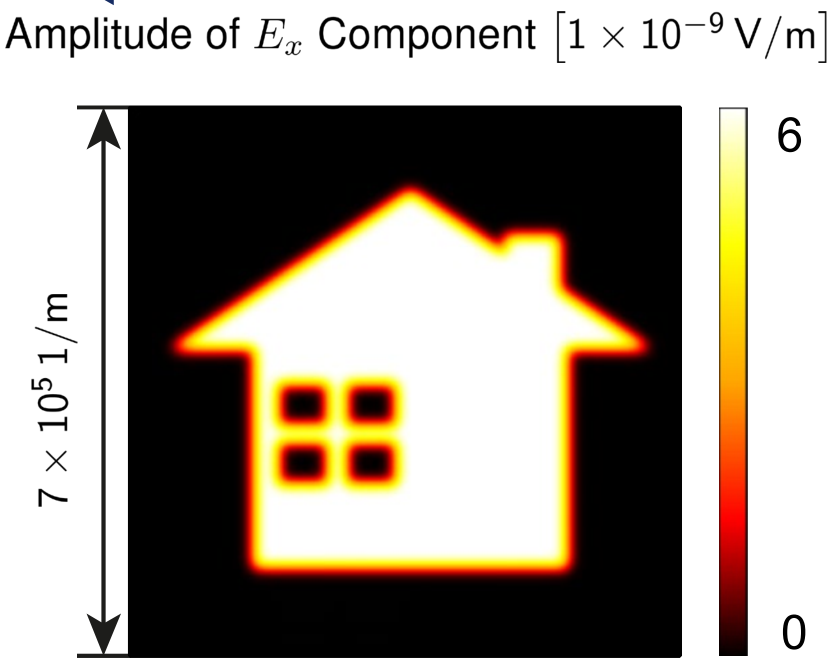
2

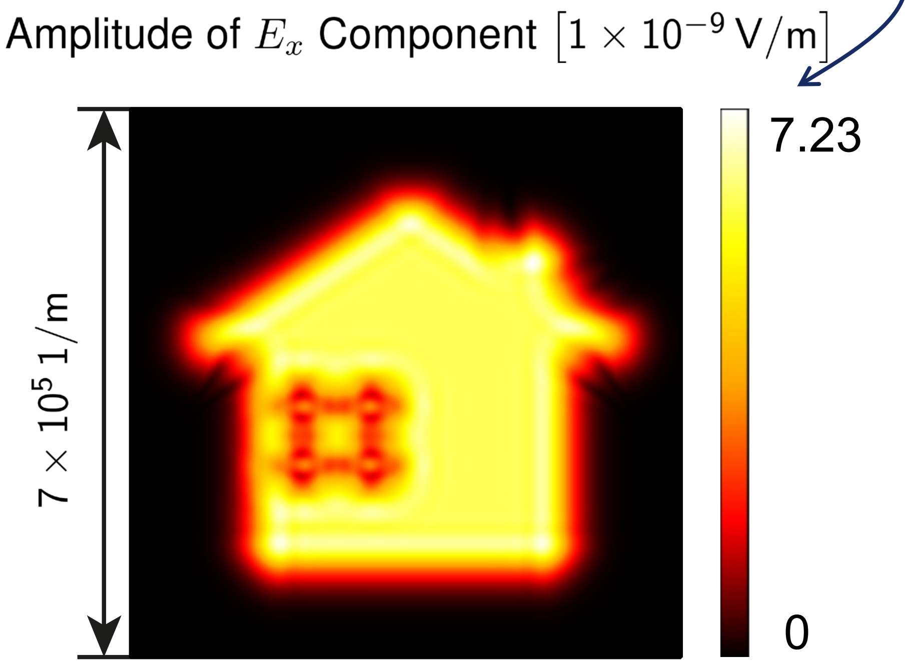


r = 50 mm,

D = 2.7 mm

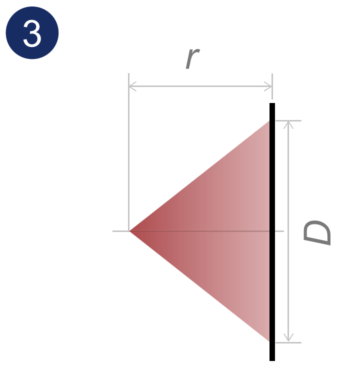
➔ NA = 0.027





HFT vs. FFT @ NA = 0.134

Sampling Points for Different Fourier Transform Techniques		
	HFT	FFT
NA = 0.134	(73 x 59)	(1333 x 925)



r = 10 mm,

D = 2.7 mm

➔ NA = 0.134

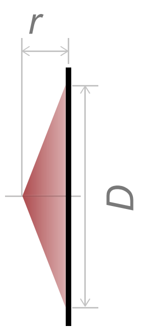
Amplitude of E_x Component $\left[1 \times 10^{-9}\,\mathrm{V/m}\right]$

Amplitude of E_x Component $\left[1 \times 10^{-9}\,\mathrm{V/m}\right]$

HFT vs. FFT @ NA = 0.804

Sampling Points for Different Fourier Transform Techniques		
	HFT	**FFT**
NA = 0.804	(73 x 59)	(7947 x 6225)

4

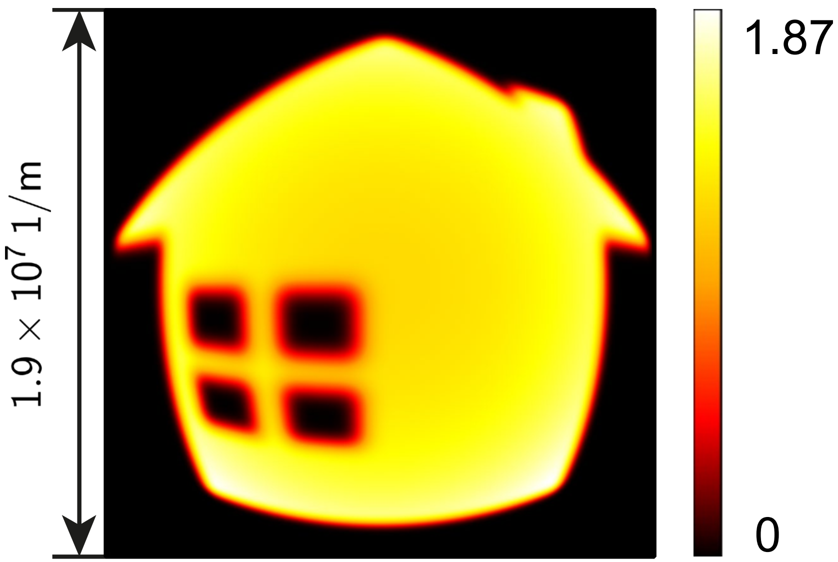


r = 1 mm,

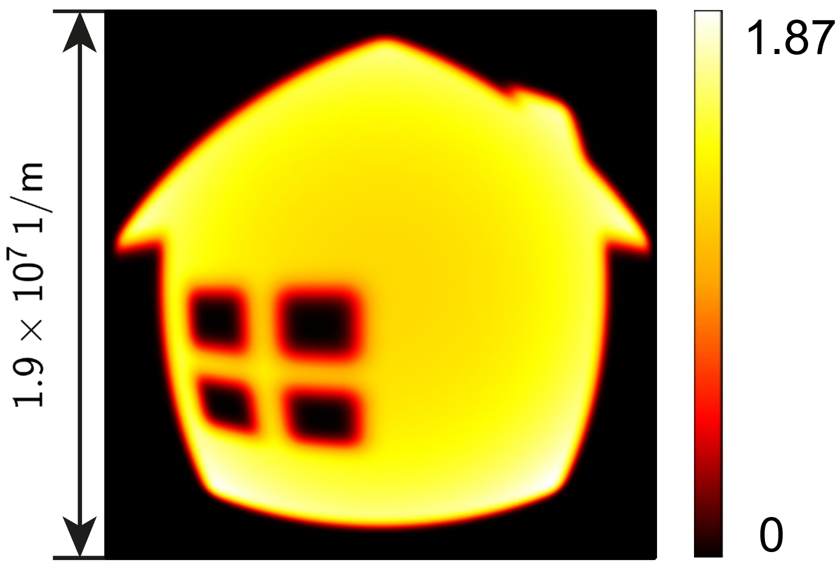
D = 2.7 mm

➔ NA = 0.804

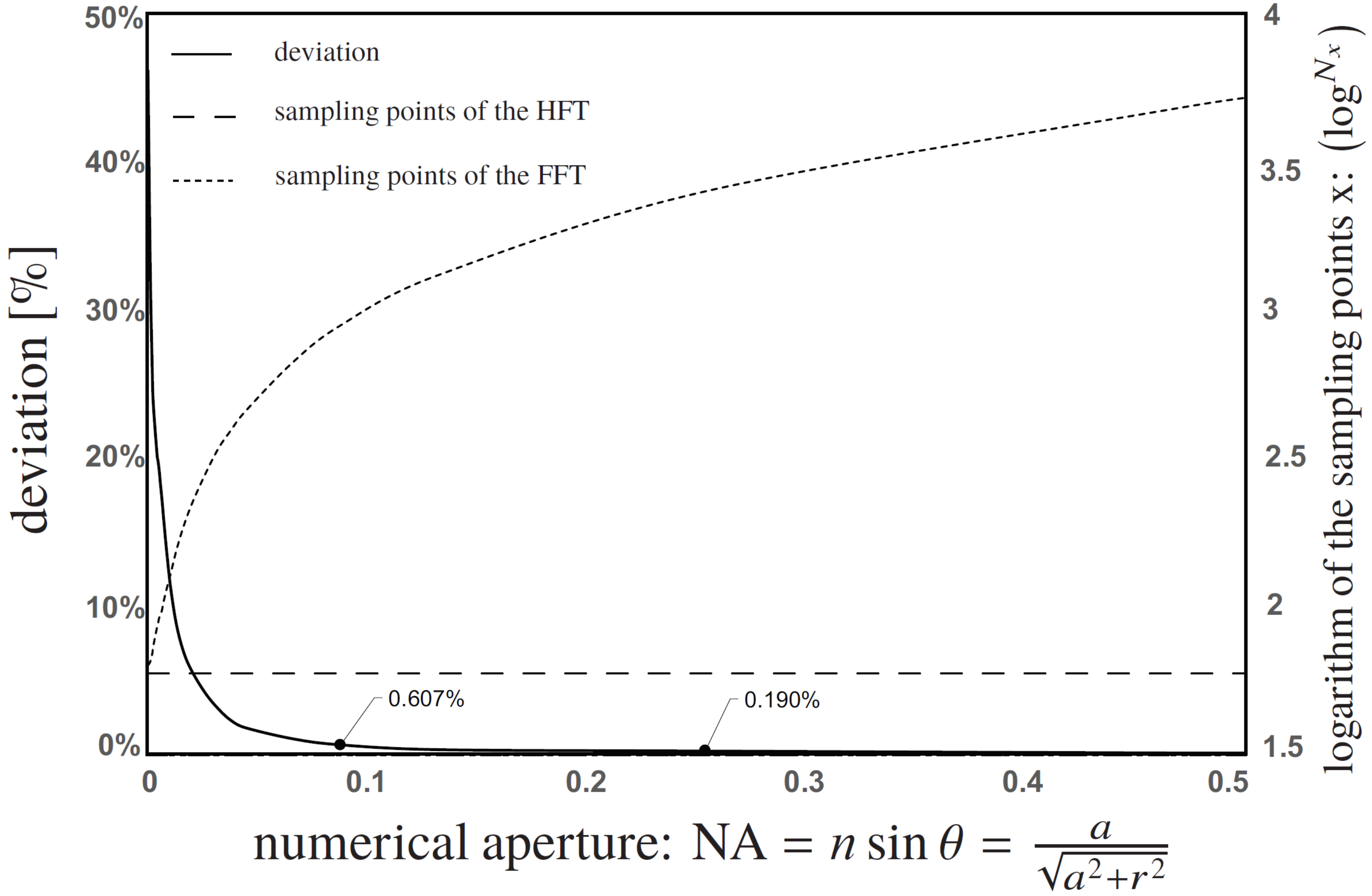
Amplitude of E_x Component $\left[1 \times 10^{-10}\,\mathrm{V/m}\right]$



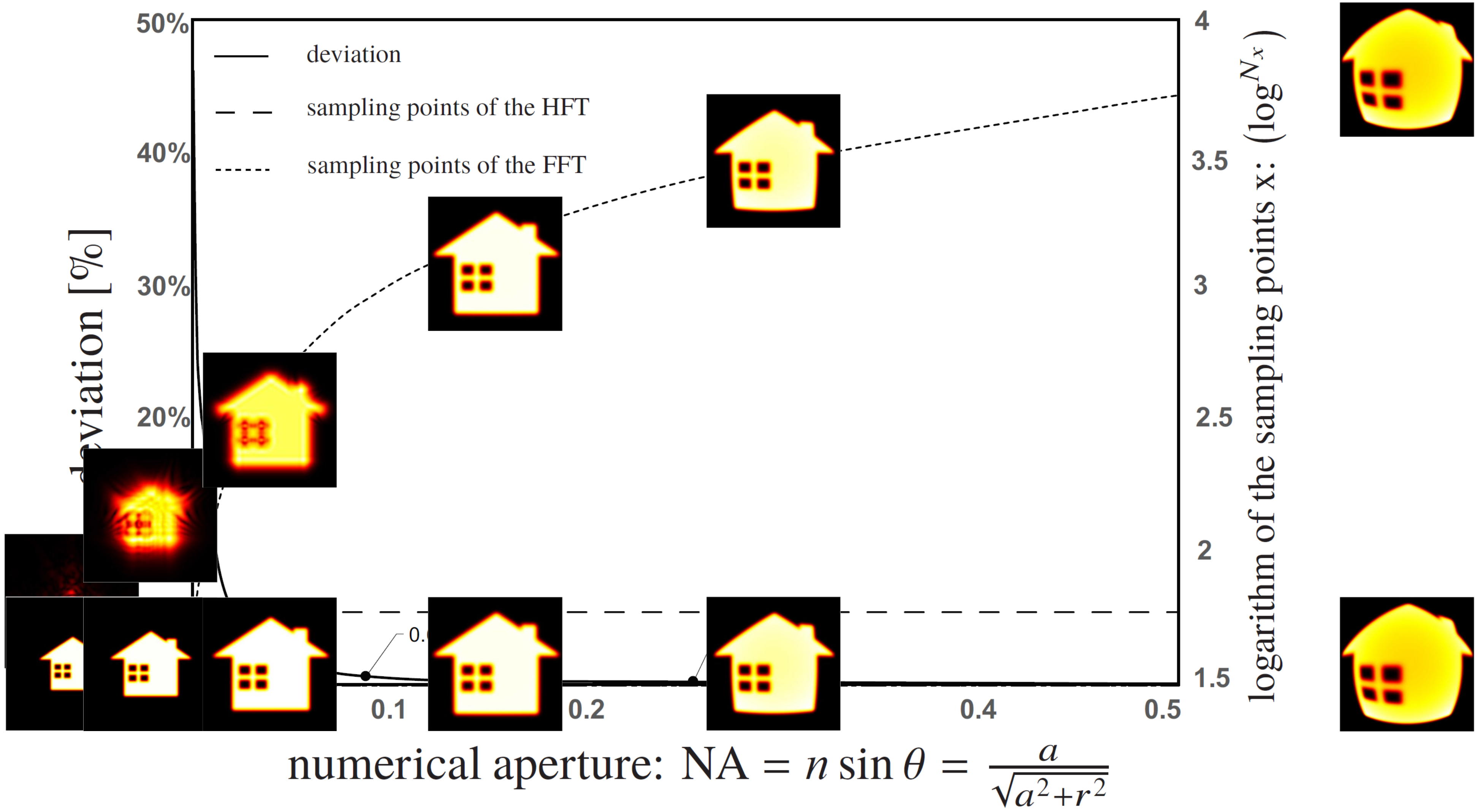
Amplitude of E_x Component $\left[1 \times 10^{-10}\,\mathrm{V/m}\right]$



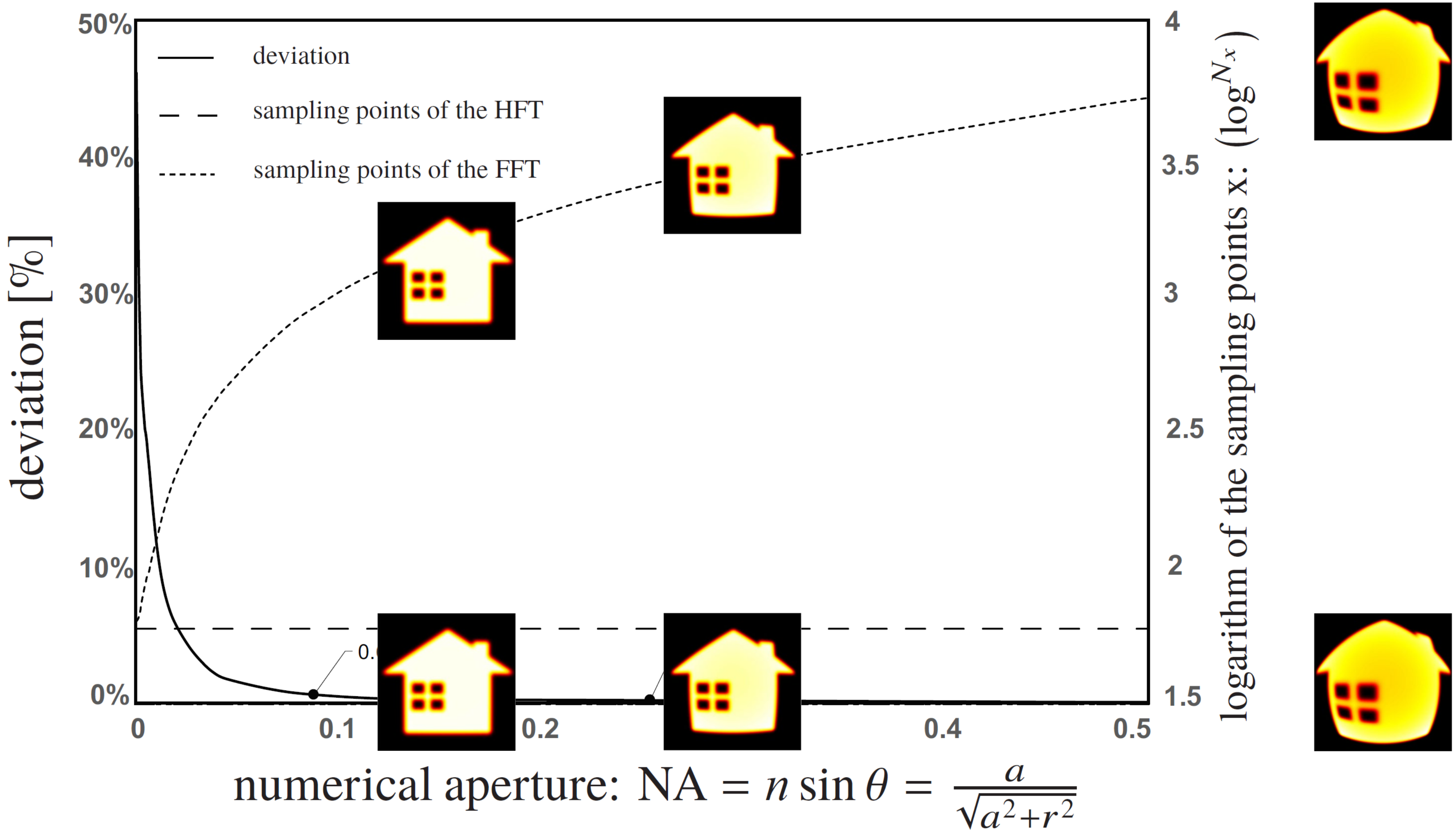
Comparison of the Numerical Effort and Accuracy



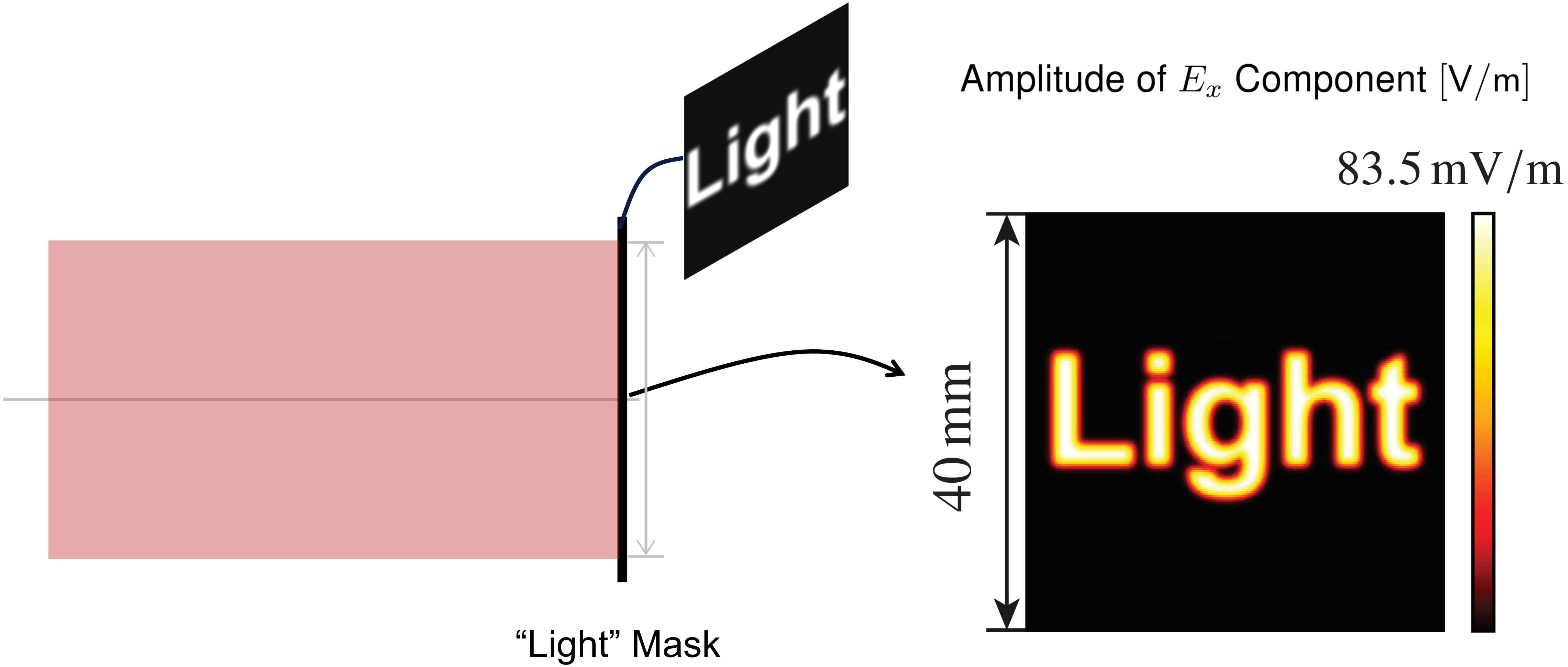
Comparison of the Numerical Effort and Accuracy



Comparison of the Numerical Effort and Accuracy

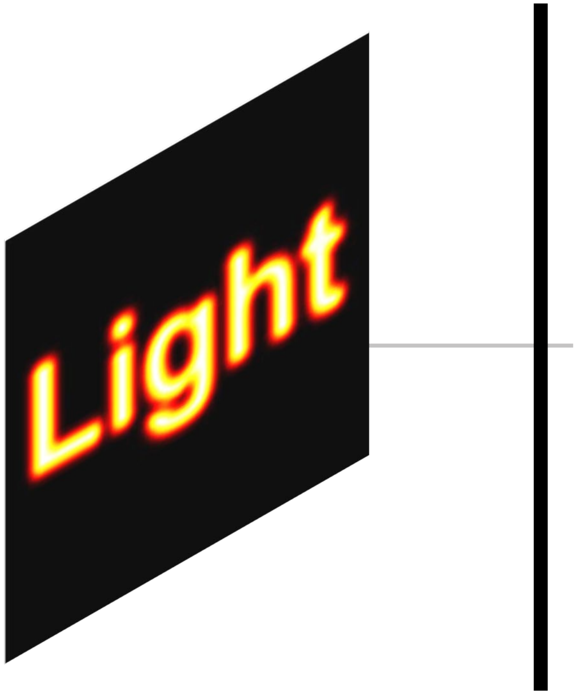


Plane Wave Illuminates the “Light” Mask



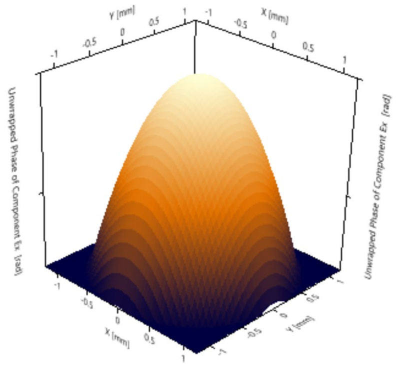
Through Zernike Phase Plate

Zernike Phase Plate

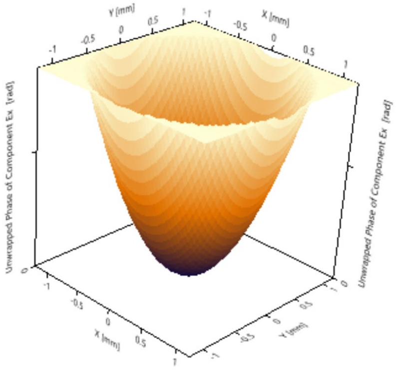


$$\psi(\boldsymbol{\rho}) = \psi^{\text{zer}}(\boldsymbol{\rho}) = k \sum_{m,n=0}^{M,N} c_n^m Z_n^m(r,\theta)$$

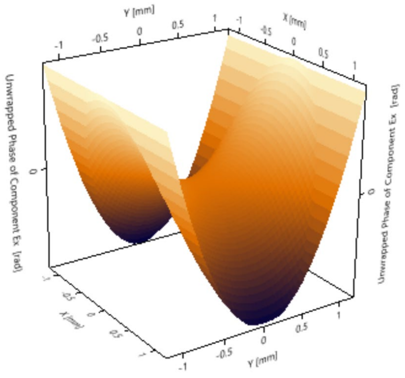
where $k = \check{n}\frac{2\pi}{\lambda}$, $r = \frac{\boldsymbol{\rho}}{|\boldsymbol{\rho}_{\max}|}$ and $\theta = \arctan(\frac{y}{x})$



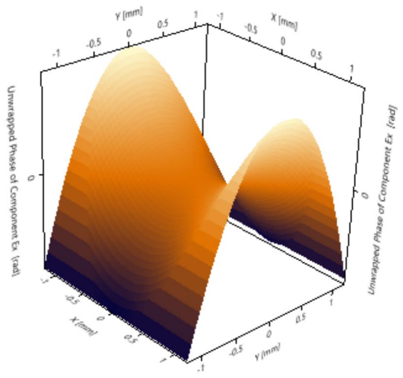
"-" defocus



"+" defocus



"+" vertical astigmatism

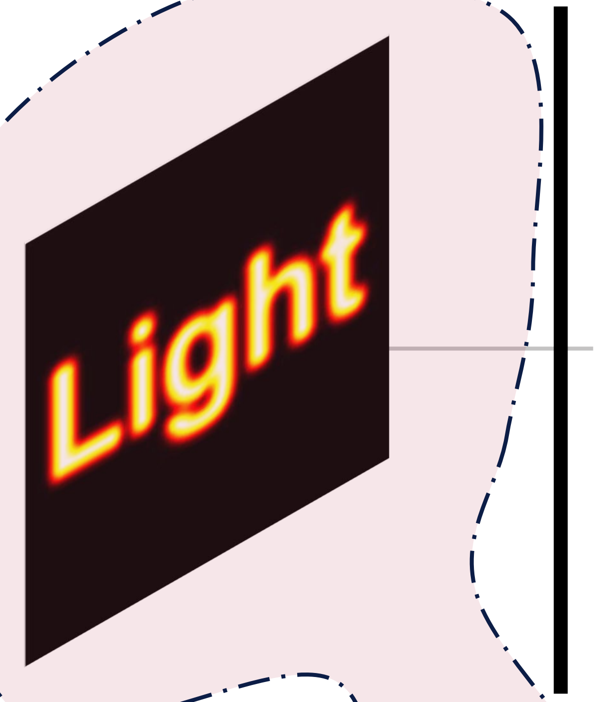


"-" vertical astigmatism

$$V_\ell(\boldsymbol{\rho}) = |V_\ell(\boldsymbol{\rho})| \exp(\mathrm{i}\psi(\boldsymbol{\rho}))$$

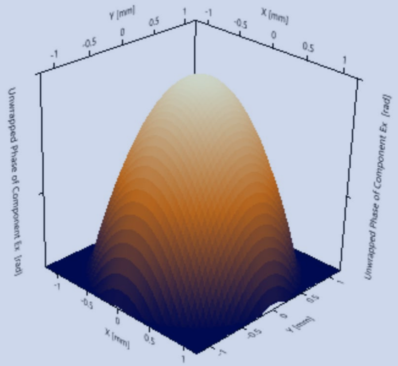
Through Zernike Phase Plate

Zernike Phase Plate

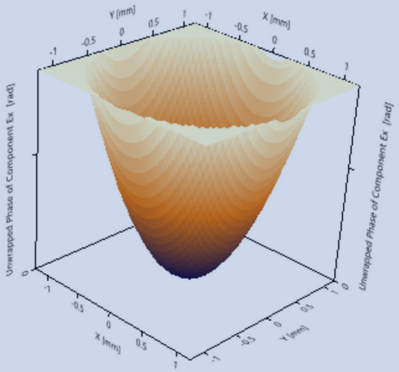


$$\psi(\boldsymbol{\rho}) = \psi^{\text{zer}}(\boldsymbol{\rho}) = k \sum_{m,n=0}^{M,N} c_n^m Z_n^m(r,\theta)$$

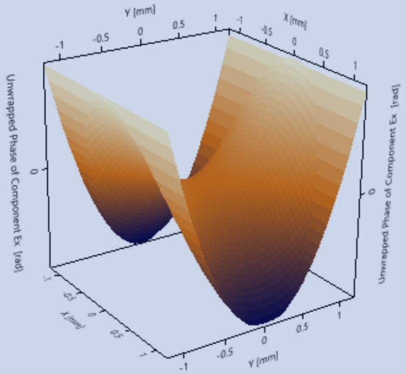
where $k = \check{n}\frac{2\pi}{\lambda}$, $r = \frac{\boldsymbol{\rho}}{|\boldsymbol{\rho}_{\text{max}}|}$ and $\theta = \arctan(\frac{y}{x})$



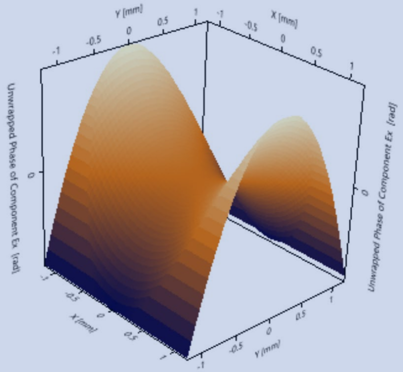
"-" defocus



"+" defocus



"+" vertical astigmatism

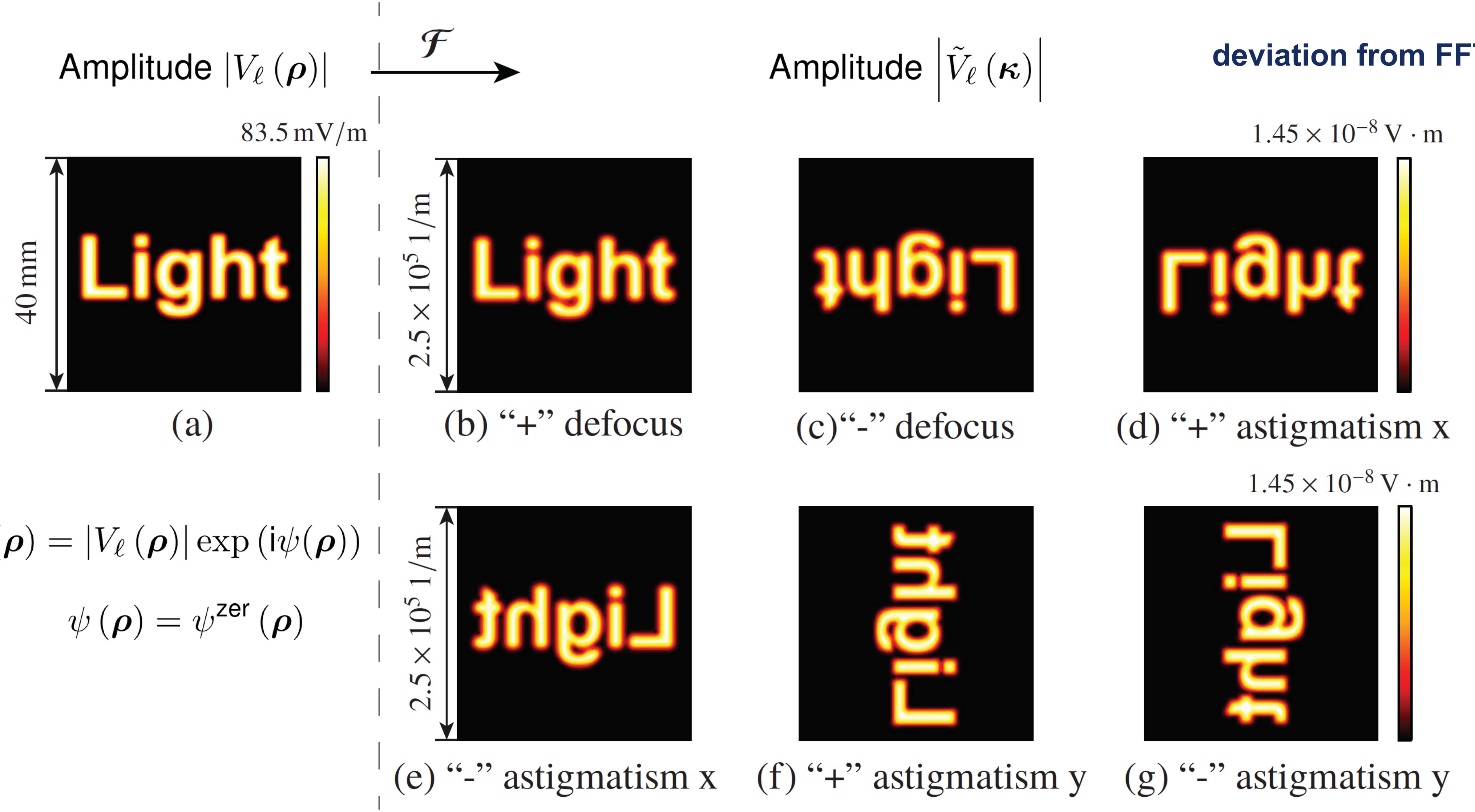


"-" vertical astigmatism

$$V_\ell(\boldsymbol{\rho}) = |V_\ell(\boldsymbol{\rho})| \exp(\mathrm{i}\psi(\boldsymbol{\rho}))$$

Results of Fourier Transform

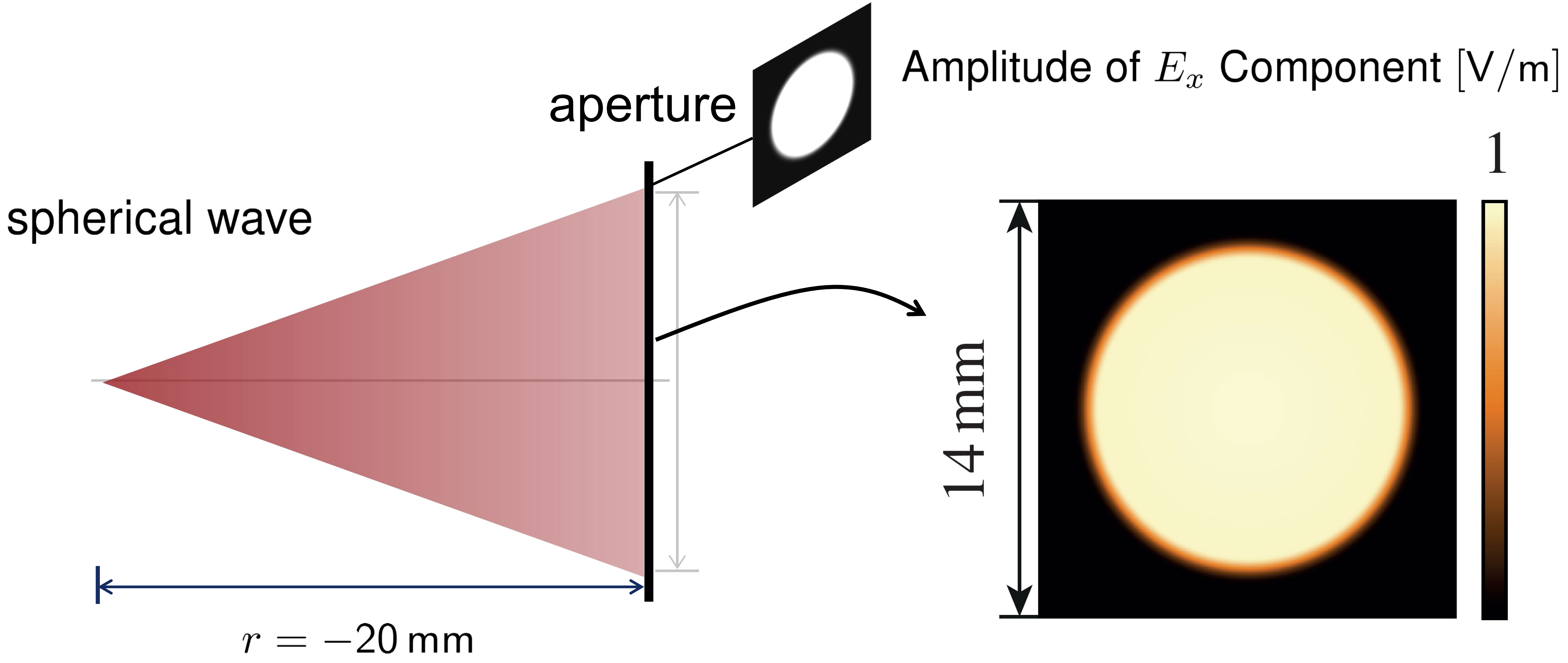
deviation from FFT ~ 0.2%



$$V_\ell(\boldsymbol{\rho}) = |V_\ell(\boldsymbol{\rho})| \exp(\mathrm{i}\psi(\boldsymbol{\rho}))$$

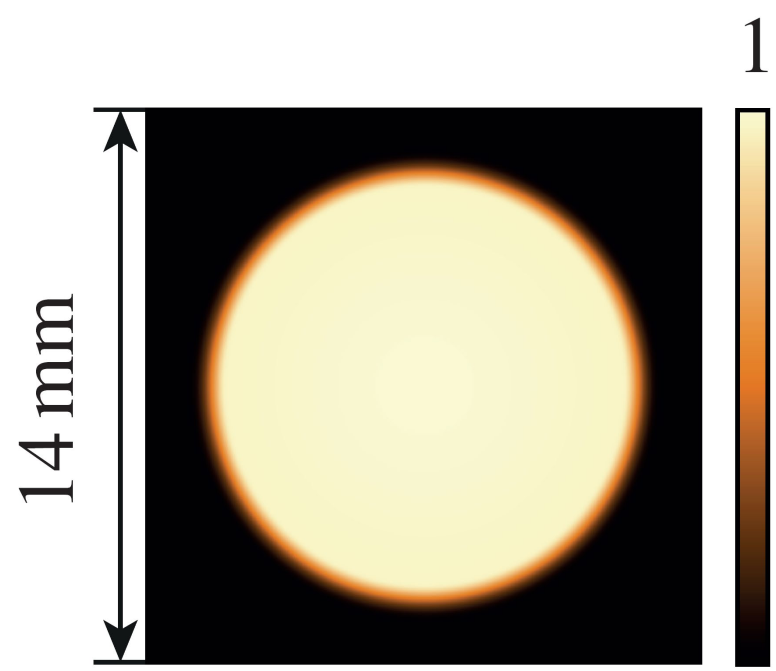
$$\psi(\boldsymbol{\rho}) = \psi^{\mathrm{zer}}(\boldsymbol{\rho})$$

Truncated Spherical Wave



Results of Fourier Transform

Amplitude of E_x Component $[\mathrm{V/m}]$

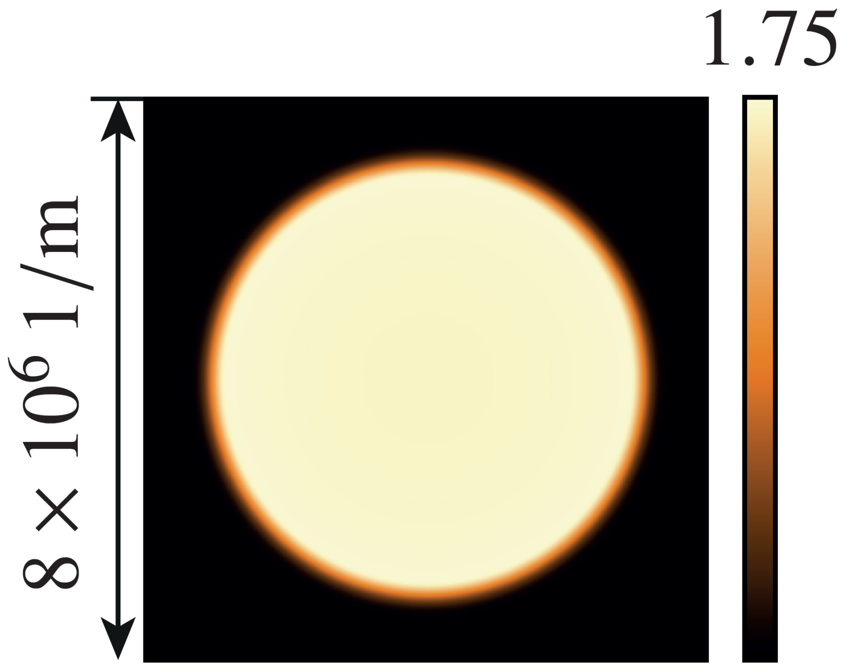


$$V_\ell(\boldsymbol{\rho}) = |V_\ell(\boldsymbol{\rho})| \exp(\mathrm{i}\psi(\boldsymbol{\rho}))$$

$$\psi(\boldsymbol{\rho}) = \psi^{\mathsf{sph}}(\boldsymbol{\rho}) = \mathsf{sgn}(r)\, k_0 \check{n} \sqrt{\boldsymbol{\rho}^2 + r^2}$$

$$\xrightarrow{\boldsymbol{\mathcal{F}}_{\kappa}}$$

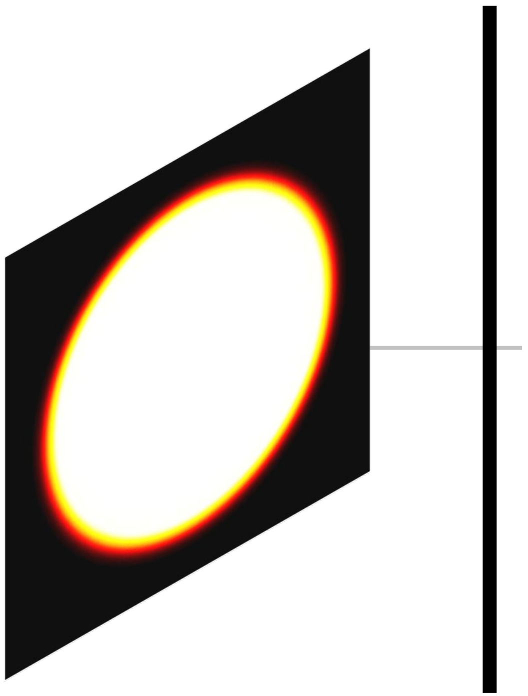
Amplitude of E_x Component $[1 \times 10^{-9}\,\mathrm{V \cdot m}]$



$$\tilde{V}_\ell(\boldsymbol{\kappa})$$

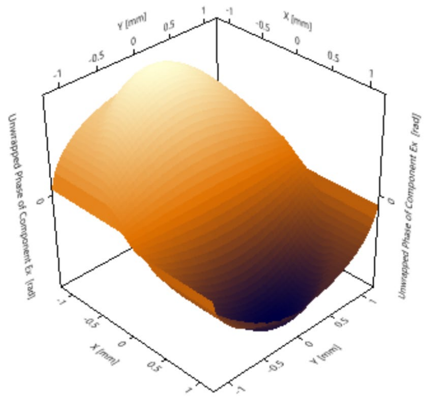
Through Zernike Phase Plate

$$V_\ell\left(\boldsymbol{\rho}\right) = \left|V_\ell\left(\boldsymbol{\rho}\right)\right| \exp\left(\mathrm{i}\psi(\boldsymbol{\rho})\right)$$

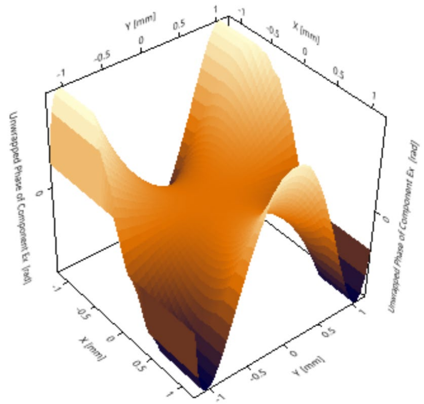


Zernike Phase Plate

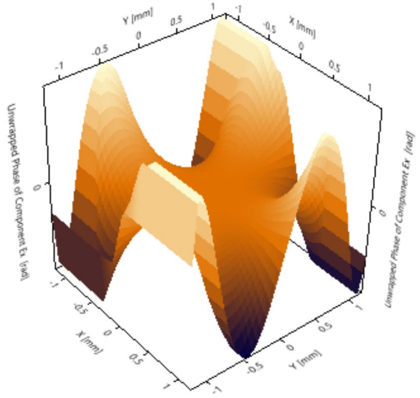
$$\psi(\boldsymbol{\rho}) = \psi^{\mathsf{sph}}(\boldsymbol{\rho}) + \psi^{\mathsf{zer}}(\boldsymbol{\rho})$$



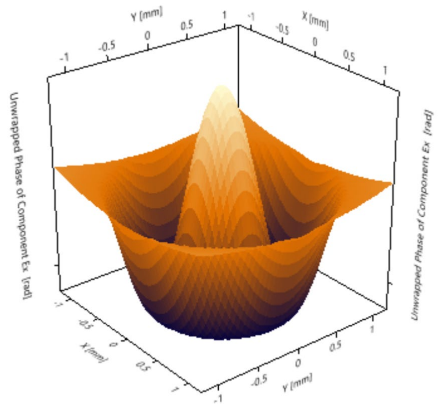
“+” coma x



“+” trefoil x



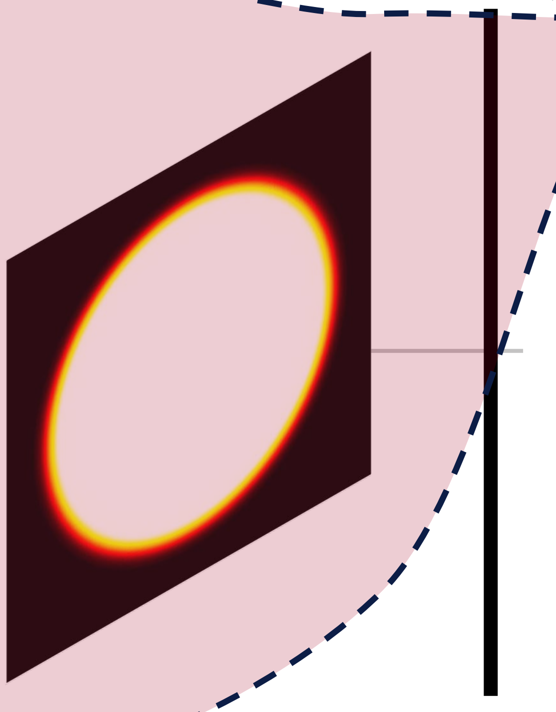
“+” tetrafoil y



“+” tertiary spherical

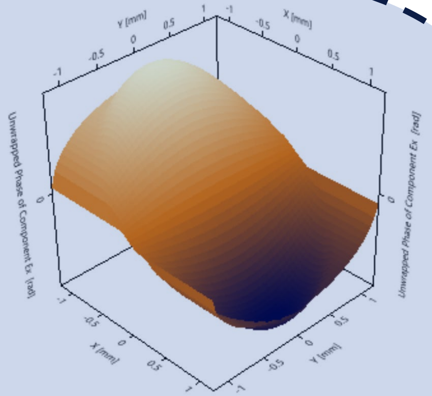
Through Zernike Phase Plate

$$V_\ell(\boldsymbol{\rho}) = |V_\ell(\boldsymbol{\rho})| \exp(\mathrm{i}\psi(\boldsymbol{\rho}))$$

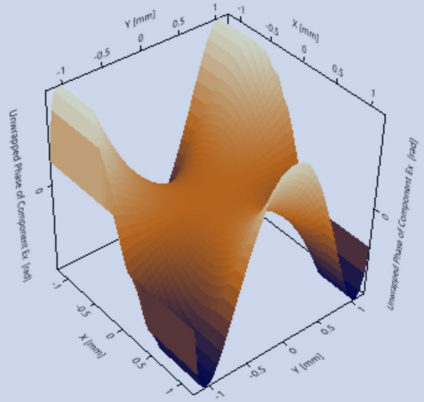


Zernike Phase Plate

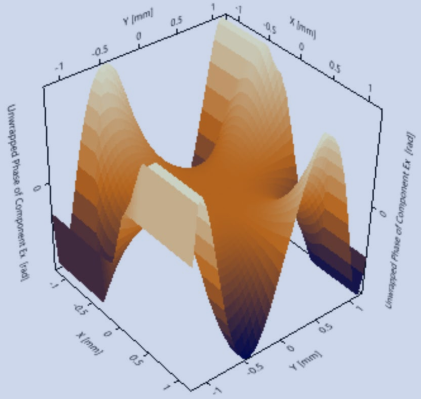
$$\psi(\boldsymbol{\rho}) = \psi^{\mathsf{sph}}(\boldsymbol{\rho}) + \psi^{\mathsf{zer}}(\boldsymbol{\rho})$$



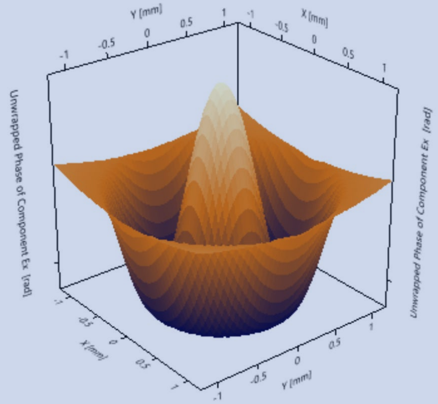
“+” coma x



“+” trefoil x



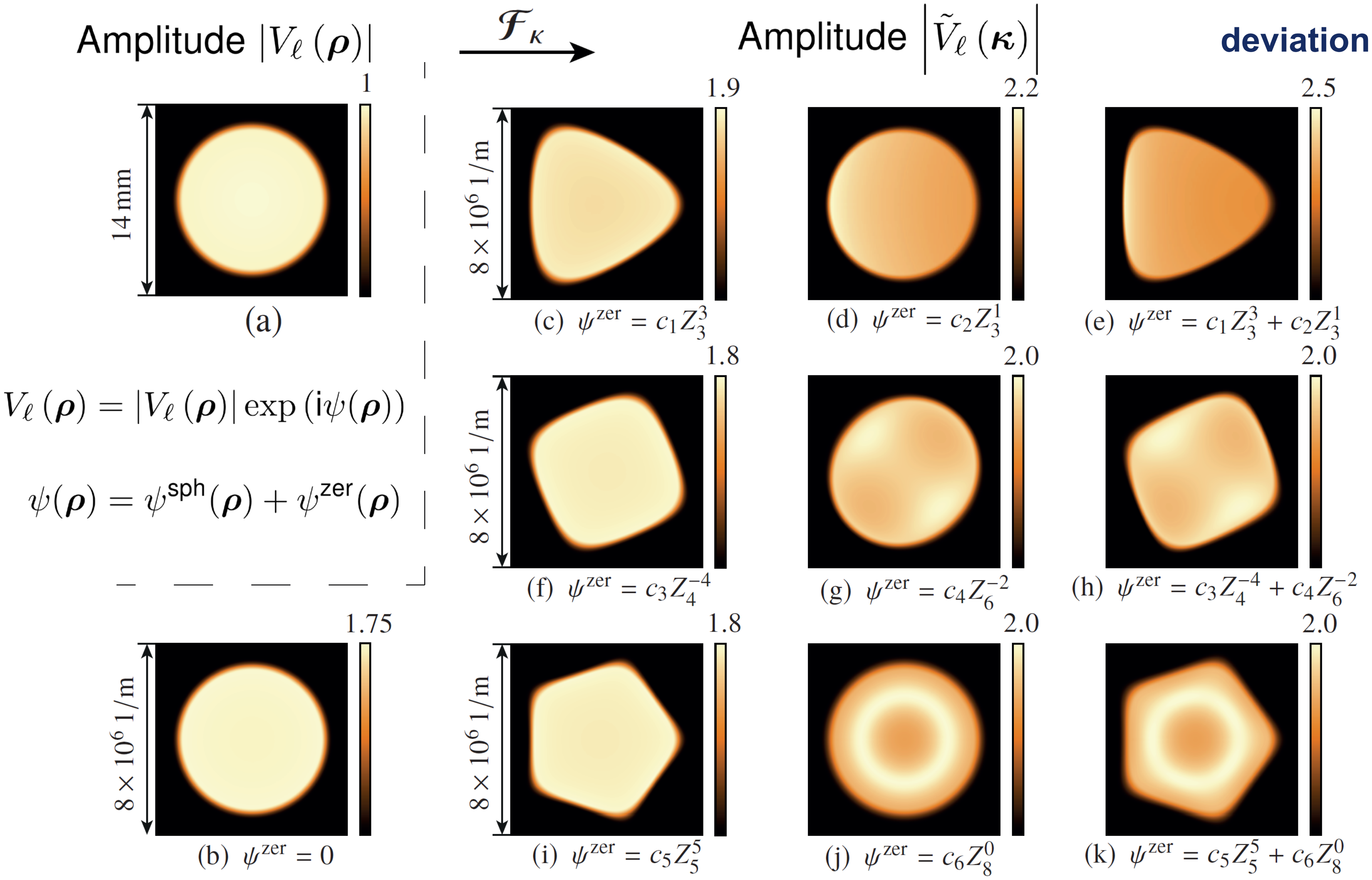
“+” tetrafoil y



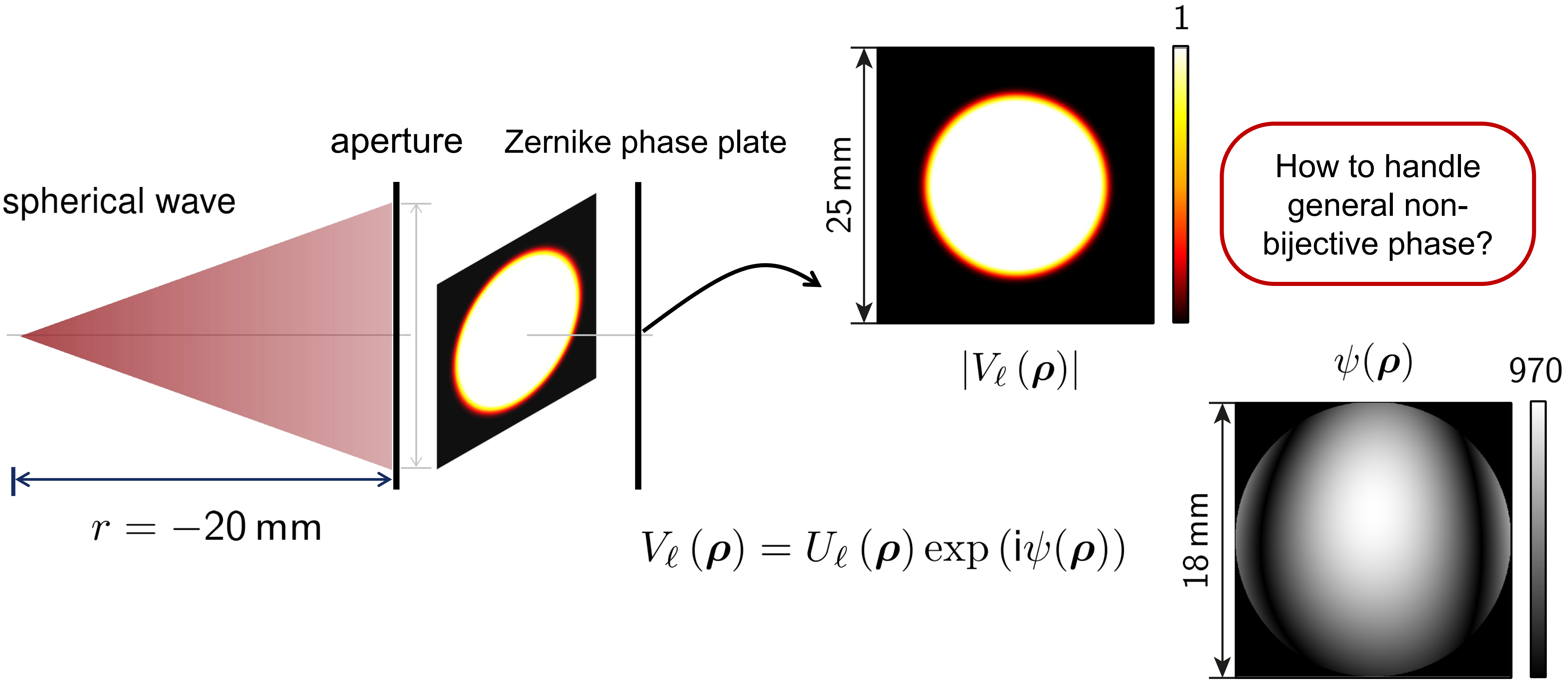
“+” tertiary spherical

Results of Fourier Transform

deviation from FFT ~ 0.1%



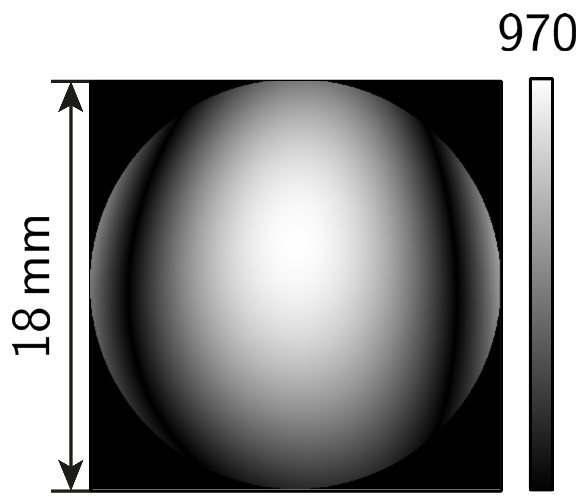
Field with Very Strong Aberrations



$$V_\ell(\boldsymbol{\rho}) = U_\ell(\boldsymbol{\rho}) \exp(\mathrm{i}\psi(\boldsymbol{\rho}))$$

How to handle general non-bijective phase?

Handling of the General Non-bijective Wavefront Phase

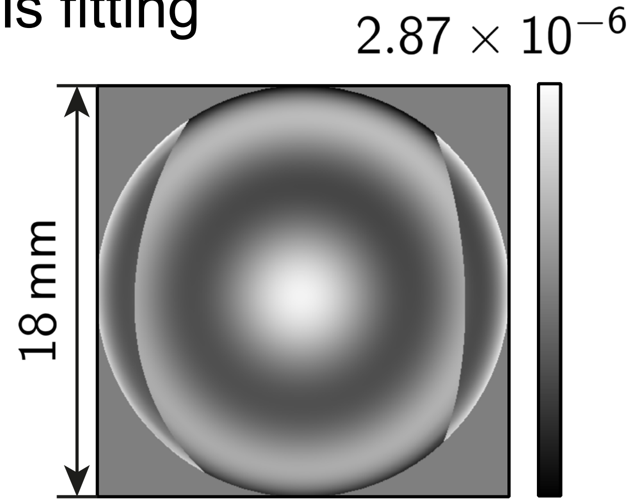


$\psi(\boldsymbol{\rho})$

$$\psi^{\text{fit}}(\boldsymbol{\rho}) = \psi^{\text{sph}}(\boldsymbol{\rho}) + \psi^{\text{zer}}(\boldsymbol{\rho})$$

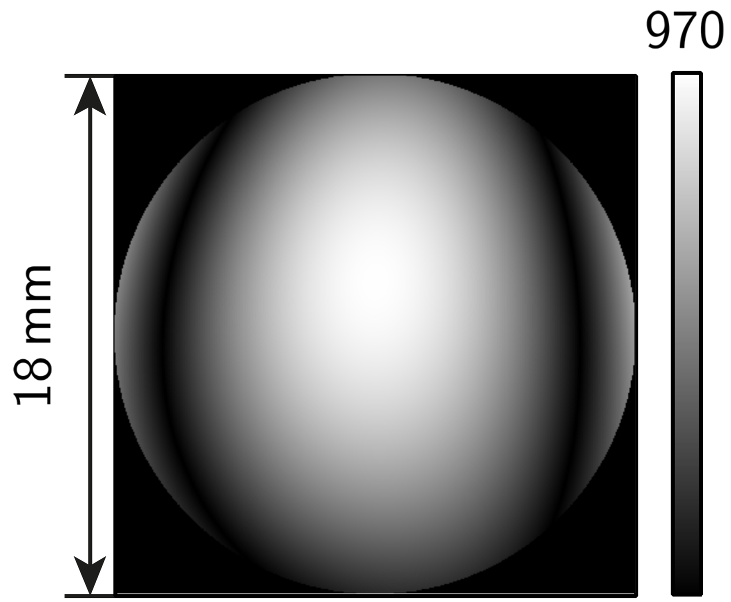
spherical phase and Zernike polynomials fitting

$$\psi(\boldsymbol{\rho}) = \psi^{\text{fit}}(\boldsymbol{\rho}) + \Delta\psi(\boldsymbol{\rho})$$

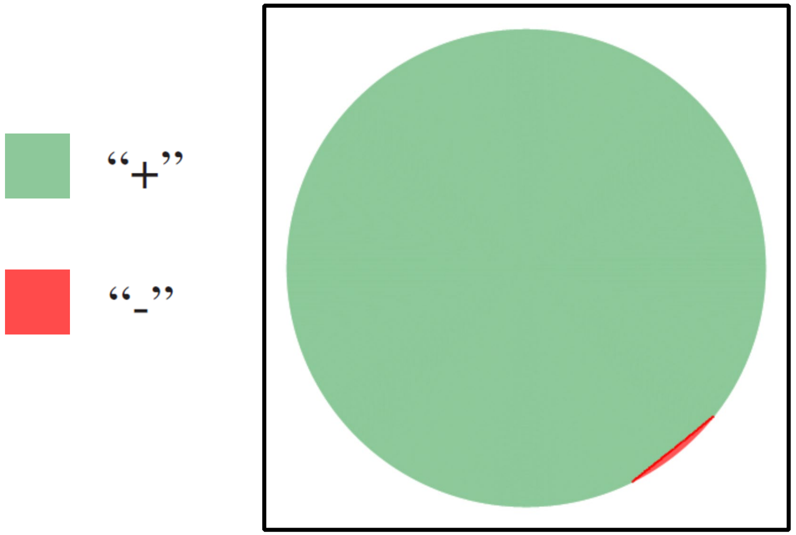


$\Delta\psi(\boldsymbol{\rho})$

Examination of the Bijectivity



$$\psi^{\text{fit}}(\boldsymbol{\rho}) = \psi^{\text{sph}}(\boldsymbol{\rho}) + \psi^{\text{zer}}(\boldsymbol{\rho})$$



sign of the $\left(\psi^{\text{fit}}_{xy}\right)^2 - \psi^{\text{fit}}_{xx}\psi^{\text{fit}}_{yy}$

criteria: the second derivative factor $\left(\psi^{\text{fit}}_{xy}\right)^2 - \psi^{\text{fit}}_{xx}\psi^{\text{fit}}_{yy}$ are all positive or all negative in the definition domain.

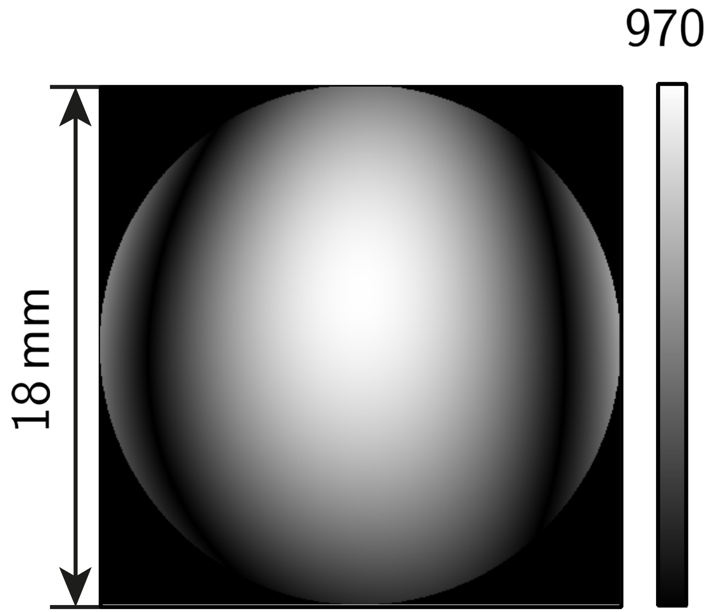
Optimization of the Non-bijective Wavefront Phase

- Elementary iterative method
  - start from the fitting result
  - omit the highest Zernike term
  - examine the bijectivity
    - Yes, stop
    - No, repeat the loop

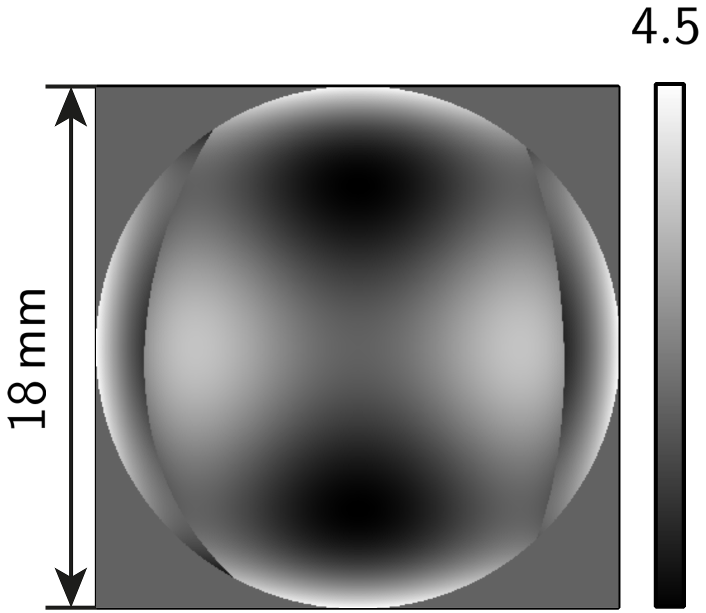
- Note: elementary optimization result might be the best, but it can be the start point of the advanced optimization method.

Name & Type	Fitting expression	Fitting coefficients	$\psi^{\mathrm{map}}(\rho)$
spherical phase	$\psi^{\mathrm{sph}} = \mathrm{sgn}(R)\, k_0 \check{n} \sqrt{\rho^2 + R^2}$	$R = 9.1$ mm	✓
Piston	$Z_0^0 = 1$	$c_0^0 = 0.88\lambda$	✓
Tilt Y	$Z_1^{-1} = 2r \sin\theta$	$c_1^{-1} = -2.52\lambda$	✓
Tilt X	$Z_1^1 = 2r \cos\theta$	$c_1^1 = -1.17\lambda$	✓
Defocus	$Z_2^0 = \sqrt{3}\,(2\rho^2 - 1)$	$c_2^0 = -1.97\lambda$	✓
Astigmatism X	$Z_2^2 = \sqrt{6} r^2 \cos 2\theta$	$c_2^2 = 27.6\lambda$	✓
Trefoil Y	$Z_3^{-3} = \sqrt{8} r^3 \sin 3\theta$	$c_3^{-3} = 6.14\lambda$	✓
Coma Y	$Z_3^{-1} = \sqrt{8}\,(3r^3 - 2r) \sin\theta$	$c_3^{-1} = 4.5\lambda$	✓
Coma X	$Z_3^1 = \sqrt{8}\,(3r^3 - 2r) \cos\theta$	$c_3^1 = 6\lambda$	✓
Trefoil X	$Z_3^3 = \sqrt{8} r^3 \cos 3\theta$	$c_3^3 = 3.84\lambda$	✓
Tetrafoil Y	$Z_4^{-4} = \sqrt{10} r^4 \sin 4\theta$	$c_4^{-4} = -3.53\lambda$	✓
Spherical aberration	$Z_4^0 = \sqrt{5}\,(6r^4 - 6r^2 + 1)$	$c_4^0 = 0.001\lambda$	✓
Secondary Astigmatism X	$Z_4^2 = \sqrt{10}\,(4r^4 - 3r^2) \cos 2\theta$	$c_4^2 = 0.15\lambda$	x

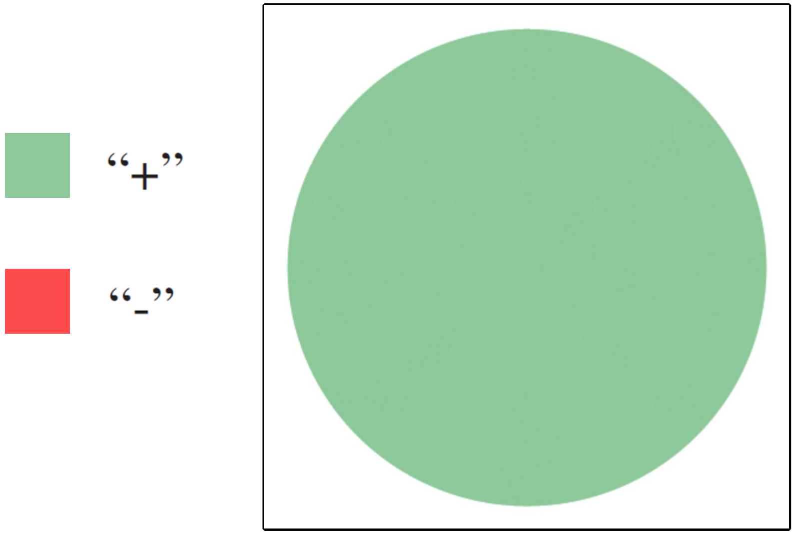
Wavefront Phase Optimization Result



$\psi(\boldsymbol{\rho})$

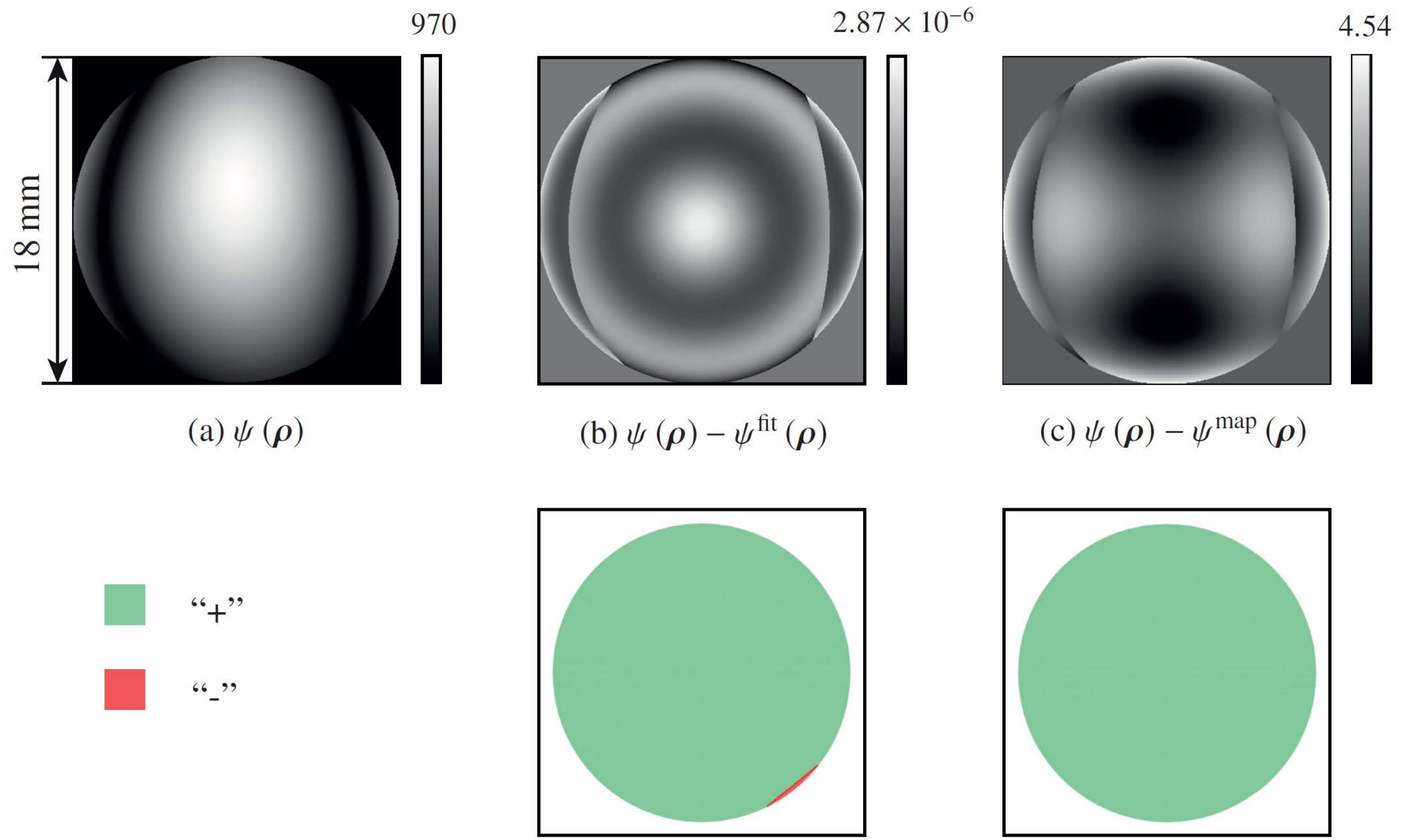


$\psi(\boldsymbol{\rho}) - \psi^{\text{map}}(\boldsymbol{\rho})$



sign of the $\left(\psi_{xy}^{\text{map}}\right)^2 - \psi_{xx}^{\text{map}}\psi_{yy}^{\text{map}}$

Handling of the General Wavefront Phase



(a) $\psi(\boldsymbol{\rho})$ (b) $\psi(\boldsymbol{\rho}) - \psi^{\text{fit}}(\boldsymbol{\rho})$ (c) $\psi(\boldsymbol{\rho}) - \psi^{\text{map}}(\boldsymbol{\rho})$

(e) sign of the $\left(\psi_{xy}^{\text{fit}}\right)^2 - \psi_{xx}^{\text{fit}}\psi_{yy}^{\text{fit}}$ (f) sign of the $\left(\psi_{xy}^{\text{map}}\right)^2 - \psi_{xx}^{\text{map}}\psi_{yy}^{\text{map}}$

1. Spherical phase and Zernike phase fitting
2. Examine the non-bijectivity
3. Optimization of the phase fitting result

Homeomorphic Fourier Transform for the Non-bijective Field

$$V_\ell(\boldsymbol{\rho}) = U_\ell(\boldsymbol{\rho}) \exp(\mathrm{i}\psi(\boldsymbol{\rho}))$$

$$\boldsymbol{\nabla}\psi(\boldsymbol{\rho}) = \boldsymbol{\kappa} \qquad (\boldsymbol{\rho} \to \boldsymbol{\kappa}, \text{ 1:1 mapping relation})$$

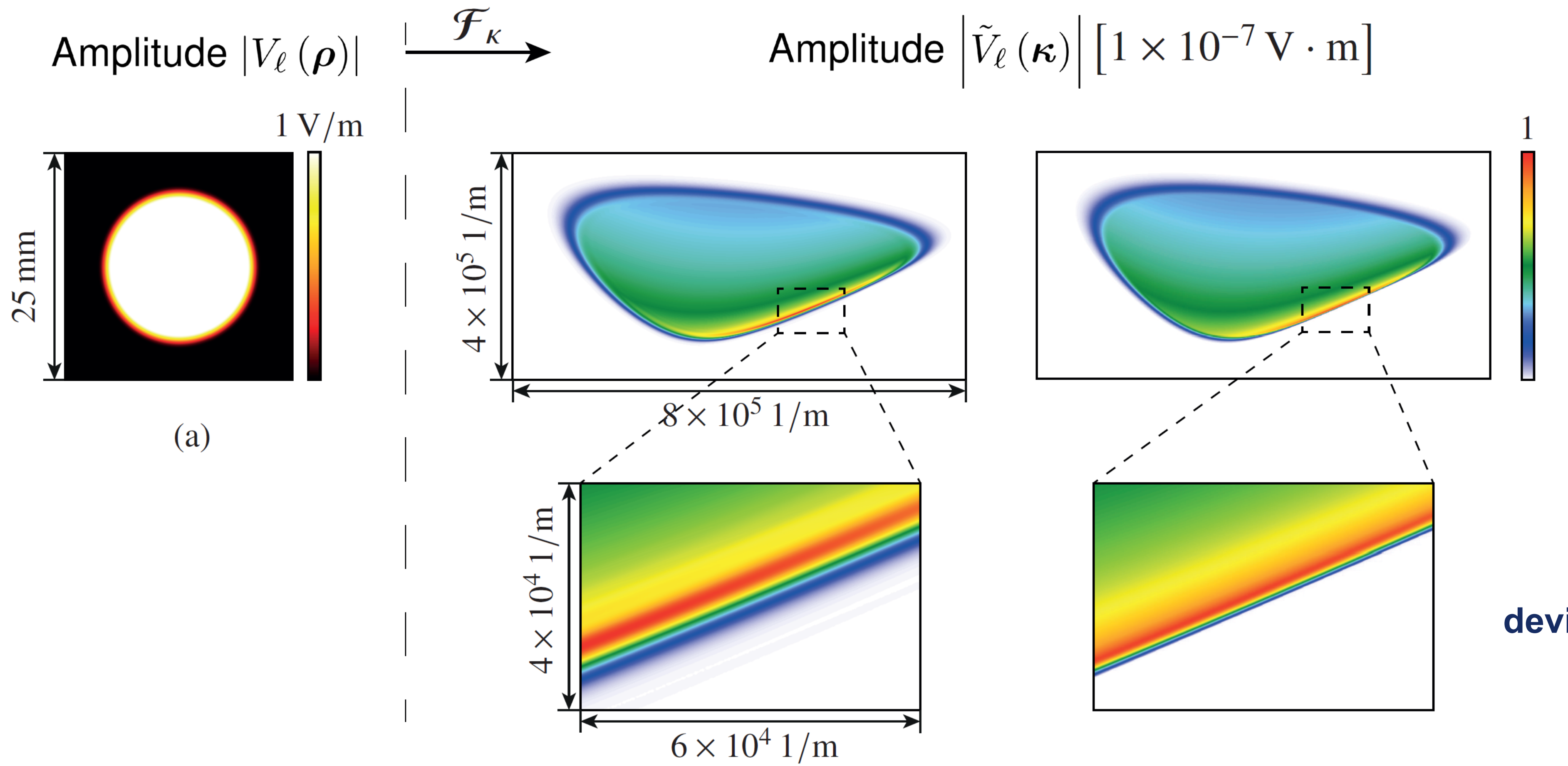
$$\tilde{V}_\perp(\boldsymbol{\kappa}) = \alpha[\boldsymbol{\rho}(\boldsymbol{\kappa})]\, U_\ell[\boldsymbol{\rho}(\boldsymbol{\kappa})] \exp(\mathrm{i}\psi[\boldsymbol{\rho}(\boldsymbol{\kappa})] - \boldsymbol{\kappa}\cdot\boldsymbol{\rho}(\boldsymbol{\kappa}))$$

$$V_\ell(\boldsymbol{\rho}) = U_\ell(\boldsymbol{\rho}) \exp(\mathrm{i}\psi^{\text{res}}(\boldsymbol{\rho})) \exp(\mathrm{i}\psi^{\text{map}}(\boldsymbol{\rho}))$$

$$\boldsymbol{\nabla}\psi^{\text{map}}(\boldsymbol{\rho}) = \boldsymbol{\kappa}^{\text{map}} \qquad (\boldsymbol{\rho} \to \boldsymbol{\kappa}^{\text{map}}, \text{ 1:1 mapping relation})$$

$$\tilde{V}_\perp(\boldsymbol{\kappa}^{\text{map}}) = \alpha[\boldsymbol{\rho}(\boldsymbol{\kappa}^{\text{map}})]\, U_\ell[\boldsymbol{\rho}(\boldsymbol{\kappa}^{\text{map}})] \exp(\mathrm{i}\psi[\boldsymbol{\rho}(\boldsymbol{\kappa}^{\text{map}})] - \boldsymbol{\kappa}^{\text{map}}\cdot\boldsymbol{\rho}(\boldsymbol{\kappa}^{\text{map}}))$$

Results of Fourier Transform



(a)

(b) FFT

(c) HFT with $\psi^{\mathrm{map}}(\boldsymbol{\rho})$

deviation from FFT ~ 1%

Summary

- Hybrid sampling strategy
  - diffractive field and smooth wavefront phase
  - reduce the number of sampling points for the 2-pi modulo phase
- Homeomorphic Fourier transform
  - stationary phase method
  - mapping type operation: N (much fewer than FFT)
- Validity of the homeomorphic Fourier transform
  - diffractive field is slow varying
  - wavefront phase is bijective (criteria: second derivative factor)
- Application of the HFT on the field with non-bijective phase
  - extract the bijective phase part
  - using new the mapping relation

Thank you for your attention!