OASIS 2019-04-01

A K-Domain Method for Fast Propagation of Electromagnetic Fields through Graded-Index Media

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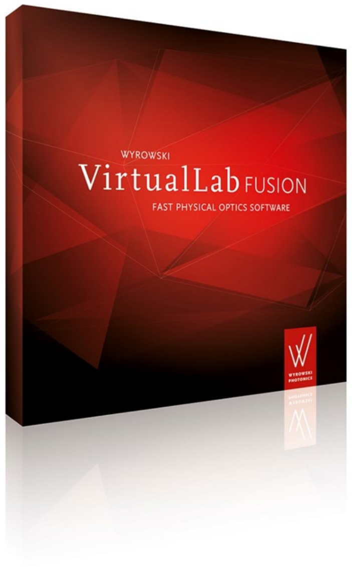
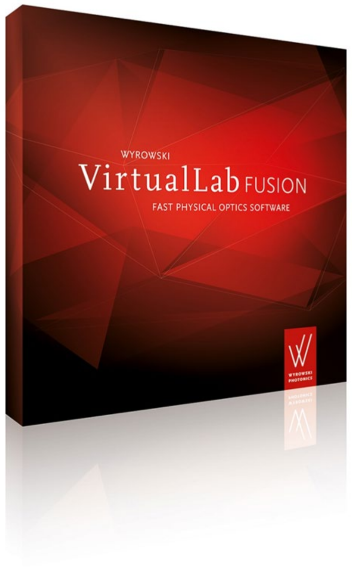
Jena, Germany



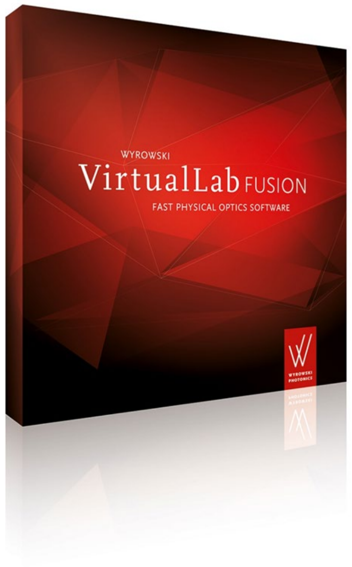
University of Jena

Applied Computational Optics Group R&D in optical modeling and design with emphasis on physical optics



Wyrowski Photonics

Wyrowski Photonics
Development of fast physical optics software VirtualLab Fusion



LightTrans International

LightTrans

- Distribution of VirtualLab Fusion, together with distributors worldwide
- Technical support, seminars, and trainings
- Engineering projects

Optical Design Software and Services

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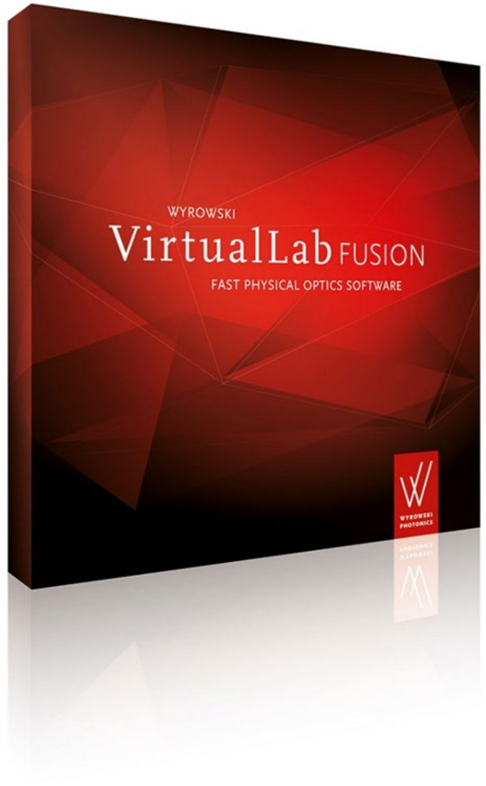
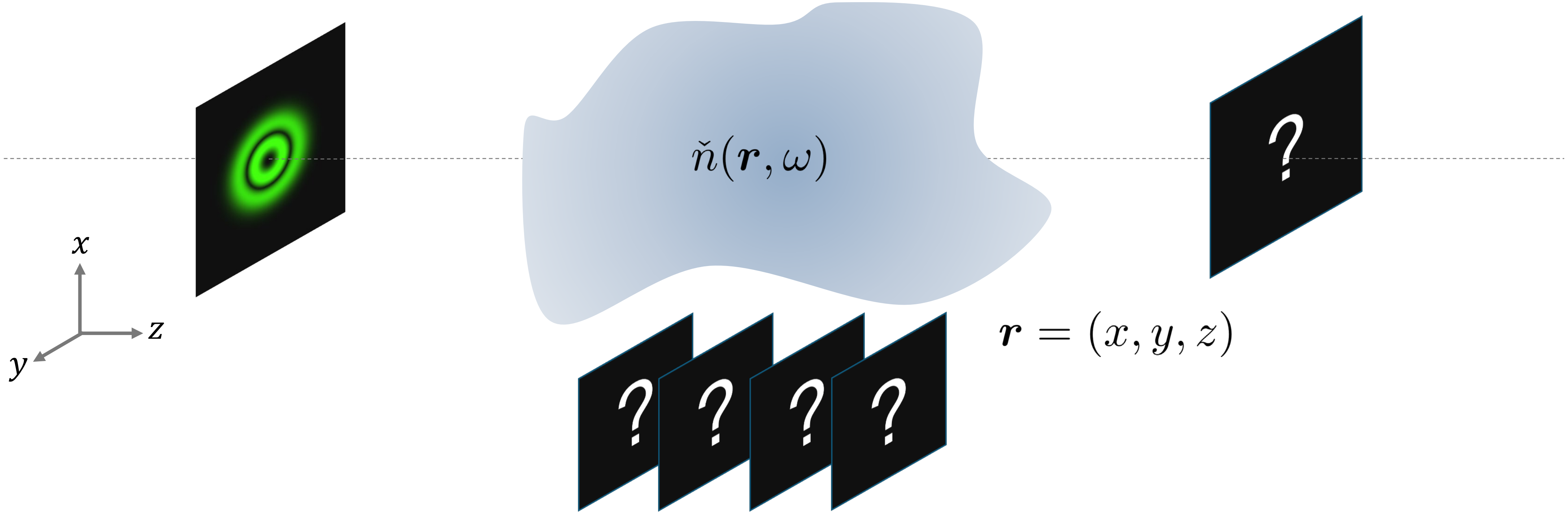


Task Description



How to calculate an electromagnetic field progation though graded-index media?

A K-Domain Method for Fast Propagation of Electromagnetic Fields through Graded-Index Media

Theory: Maxwell‘s Equations

$$\nabla \times \boldsymbol{E}(\boldsymbol{r},\omega) = i\omega\mu_0\boldsymbol{H}(\boldsymbol{r},\omega) \tag{1}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r},\omega) = i\omega\epsilon(\boldsymbol{r},\omega)\boldsymbol{E}(\boldsymbol{r},\omega) \tag{2}$$

$$\epsilon(\boldsymbol{r},\omega) = \check{n}^2(\boldsymbol{r},\omega)$$

Now we define $\boldsymbol{V}(\boldsymbol{r},\omega) = \{E_x, E_y, E_z, \sqrt{\frac{\mu_0}{\epsilon_0}}H_x, \sqrt{\frac{\mu_0}{\epsilon_0}}H_y, \sqrt{\frac{\mu_0}{\epsilon_0}}H_z\}^T(\boldsymbol{r},\omega)$. Then Eqn. (1) and (2) can be rewritten as

ω is skipped in notation.

$$\begin{pmatrix} \partial_y V_3(\boldsymbol{r}) - \partial_z V_2(\boldsymbol{r}) \\ \partial_z V_1(\boldsymbol{r}) - \partial_x V_3(\boldsymbol{r}) \\ \partial_x V_2(\boldsymbol{r}) - \partial_y V_1(\boldsymbol{r}) \end{pmatrix} = \mathrm{i}k_0 \begin{pmatrix} V_4(\boldsymbol{r}) \\ V_5(\boldsymbol{r}) \\ V_6(\boldsymbol{r}) \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} \partial_y V_6(\boldsymbol{r}) - \partial_z V_5(\boldsymbol{r}) \\ \partial_z V_4(\boldsymbol{r}) - \partial_x V_6(\boldsymbol{r}) \\ \partial_x V_5(\boldsymbol{r}) - \partial_y V_4(\boldsymbol{r}) \end{pmatrix} = -\mathrm{i}k_0\epsilon(\boldsymbol{r}) \begin{pmatrix} V_1(\boldsymbol{r}) \\ V_2(\boldsymbol{r}) \\ V_3(\boldsymbol{r}) \end{pmatrix} \tag{4}$$

Theory: Fourier Transform

$$\begin{pmatrix} \partial_y V_3(\boldsymbol{r}) - \partial_z V_2(\boldsymbol{r}) \\ \partial_z V_1(\boldsymbol{r}) - \partial_x V_3(\boldsymbol{r}) \\ \partial_x V_2(\boldsymbol{r}) - \partial_y V_1(\boldsymbol{r}) \end{pmatrix} = \mathrm{i}k_0 \begin{pmatrix} V_4(\boldsymbol{r}) \\ V_5(\boldsymbol{r}) \\ V_6(\boldsymbol{r}) \end{pmatrix} \quad \begin{pmatrix} \partial_y V_6(\boldsymbol{r}) - \partial_z V_5(\boldsymbol{r}) \\ \partial_z V_4(\boldsymbol{r}) - \partial_x V_6(\boldsymbol{r}) \\ \partial_x V_5(\boldsymbol{r}) - \partial_y V_4(\boldsymbol{r}) \end{pmatrix} = -\mathrm{i}k_0 \epsilon(\boldsymbol{r}) \begin{pmatrix} V_1(\boldsymbol{r}) \\ V_2(\boldsymbol{r}) \\ V_3(\boldsymbol{r}) \end{pmatrix} \quad \text{(3-4)}$$

In the plane z, we represent $V_\ell(\boldsymbol{\rho}, z)$ by inverse Fourier transform $\boldsymbol{\rho} = (x, y)$

$$V_\ell(\boldsymbol{\rho}, z) = \mathcal{F}_k^{-1} \tilde{V}_\ell(\boldsymbol{\kappa}, z) = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \mathrm{d}k_x \, \mathrm{d}k_y \; \tilde{V}_\ell(\boldsymbol{\kappa}, z) \exp(\mathrm{i}\boldsymbol{\kappa} \cdot \boldsymbol{\rho}) \, . \quad (5)$$

And substitute into Eqn. (3) and (4), i.e.,

$\boldsymbol{\kappa} = (\kappa_x, \kappa_y)$

$$\partial_x V_\ell(\boldsymbol{\rho}, z) = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \mathrm{d}k_x \, \mathrm{d}k_y \; \mathrm{i}\kappa_x \tilde{V}_\ell(\boldsymbol{\kappa}, z) \exp(\mathrm{i}\boldsymbol{\kappa} \cdot \boldsymbol{\rho})$$

and

$$\partial_y V_\ell(\boldsymbol{\rho}, z) = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \mathrm{d}k_x \, \mathrm{d}k_y \; \mathrm{i}\kappa_y \tilde{V}_\ell(\boldsymbol{\kappa}, z) \exp(\mathrm{i}\boldsymbol{\kappa} \cdot \boldsymbol{\rho})$$

Theory: K-Domain Fomulation

$$\begin{pmatrix} \partial_y V_3(\boldsymbol{r}) - \partial_z V_2(\boldsymbol{r}) \\ \partial_z V_1(\boldsymbol{r}) - \partial_x V_3(\boldsymbol{r}) \\ \partial_x V_2(\boldsymbol{r}) - \partial_y V_1(\boldsymbol{r}) \end{pmatrix} = \mathrm{i}k_0 \begin{pmatrix} V_4(\boldsymbol{r}) \\ V_5(\boldsymbol{r}) \\ V_6(\boldsymbol{r}) \end{pmatrix} \quad \begin{pmatrix} \partial_y V_6(\boldsymbol{r}) - \partial_z V_5(\boldsymbol{r}) \\ \partial_z V_4(\boldsymbol{r}) - \partial_x V_6(\boldsymbol{r}) \\ \partial_x V_5(\boldsymbol{r}) - \partial_y V_4(\boldsymbol{r}) \end{pmatrix} = -\mathrm{i}k_0 \epsilon(\boldsymbol{r}) \begin{pmatrix} V_1(\boldsymbol{r}) \\ V_2(\boldsymbol{r}) \\ V_3(\boldsymbol{r}) \end{pmatrix} \qquad \text{(3-4)}$$

Eqn. (3) and (4) become

$$\begin{pmatrix} \mathrm{i}\kappa_y \tilde{V}_3(\boldsymbol{\kappa}, z) - \partial_z \tilde{V}_2(\boldsymbol{\kappa}, z) \\ \partial_z \tilde{V}_1(\boldsymbol{\kappa}, z) - \mathrm{i}\kappa_x \tilde{V}_3(\boldsymbol{\kappa}, z) \\ \mathrm{i}\kappa_x \tilde{V}_2(\boldsymbol{\kappa}, z) - \mathrm{i}\kappa_y \tilde{V}_1(\boldsymbol{\kappa}, z) \end{pmatrix} = \mathrm{i}k_0 \begin{pmatrix} \tilde{V}_4(\boldsymbol{\kappa}, z) \\ \tilde{V}_5(\boldsymbol{\kappa}, z) \\ \tilde{V}_6(\boldsymbol{\kappa}, z) \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{i}\kappa_y \tilde{V}_6(\boldsymbol{\kappa}, z) - \partial_z \tilde{V}_5(\boldsymbol{\kappa}, z) \\ \partial_z \tilde{V}_4(\boldsymbol{\kappa}, z) - \mathrm{i}\kappa_x \tilde{V}_6(\boldsymbol{\kappa}, z) \\ \mathrm{i}\kappa_x \tilde{V}_5(\boldsymbol{\kappa}, z) - \mathrm{i}\kappa_y \tilde{V}_4(\boldsymbol{\kappa}, z) \end{pmatrix} = -\mathrm{i}k_0 \tilde{\epsilon}(\boldsymbol{\kappa}, z) * \begin{pmatrix} \tilde{V}_1(\boldsymbol{\kappa}, z) \\ \tilde{V}_2(\boldsymbol{\kappa}, z) \\ \tilde{V}_3(\boldsymbol{\kappa}, z) \end{pmatrix}$$

$$\partial_z => \frac{\mathrm{d}}{\mathrm{d}z}$$

Theory: K-Domain Fomulation

$$\begin{pmatrix} \partial_y V_3(\boldsymbol{r}) - \partial_z V_2(\boldsymbol{r}) \\ \partial_z V_1(\boldsymbol{r}) - \partial_x V_3(\boldsymbol{r}) \\ \partial_x V_2(\boldsymbol{r}) - \partial_y V_1(\boldsymbol{r}) \end{pmatrix} = \mathrm{i}k_0 \begin{pmatrix} V_4(\boldsymbol{r}) \\ V_5(\boldsymbol{r}) \\ V_6(\boldsymbol{r}) \end{pmatrix} \quad \begin{pmatrix} \partial_y V_6(\boldsymbol{r}) - \partial_z V_5(\boldsymbol{r}) \\ \partial_z V_4(\boldsymbol{r}) - \partial_x V_6(\boldsymbol{r}) \\ \partial_x V_5(\boldsymbol{r}) - \partial_y V_4(\boldsymbol{r}) \end{pmatrix} = -\mathrm{i}k_0 \epsilon(\boldsymbol{r}) \begin{pmatrix} V_1(\boldsymbol{r}) \\ V_2(\boldsymbol{r}) \\ V_3(\boldsymbol{r}) \end{pmatrix} \qquad \text{(3-4)}$$

Eqn. (3) and (4) become

$$\begin{pmatrix} \mathrm{i}\kappa_y \tilde{V}_3(\boldsymbol{\kappa}, z) - \frac{\mathrm{d}\tilde{V}_2}{\mathrm{d}z}(\boldsymbol{\kappa}, z) \\ \frac{\mathrm{d}\tilde{V}_1}{\mathrm{d}z}(\boldsymbol{\kappa}, z) - \mathrm{i}\kappa_x \tilde{V}_3(\boldsymbol{\kappa}, z) \\ \mathrm{i}\kappa_x \tilde{V}_2(\boldsymbol{\kappa}, z) - \mathrm{i}\kappa_y \tilde{V}_1(\boldsymbol{\kappa}, z) \end{pmatrix} = \mathrm{i}k_0 \begin{pmatrix} \tilde{V}_4(\boldsymbol{\kappa}, z) \\ \tilde{V}_5(\boldsymbol{\kappa}, z) \\ \tilde{V}_6(\boldsymbol{\kappa}, z) \end{pmatrix} \qquad \text{(5)}$$

and

$$\begin{pmatrix} \mathrm{i}\kappa_y \tilde{V}_6(\boldsymbol{\kappa}, z) - \frac{\mathrm{d}\tilde{V}_5}{\mathrm{d}z}(\boldsymbol{\kappa}, z) \\ \frac{\mathrm{d}\tilde{V}_4}{\mathrm{d}z}(\boldsymbol{\kappa}, z) - \mathrm{i}\kappa_x \tilde{V}_6(\boldsymbol{\kappa}, z) \\ \mathrm{i}\kappa_x \tilde{V}_5(\boldsymbol{\kappa}, z) - \mathrm{i}\kappa_y \tilde{V}_4(\boldsymbol{\kappa}, z) \end{pmatrix} = -\mathrm{i}k_0 \tilde{\epsilon}(\boldsymbol{\kappa}, z) * \begin{pmatrix} \tilde{V}_1(\boldsymbol{\kappa}, z) \\ \tilde{V}_2(\boldsymbol{\kappa}, z) \\ \tilde{V}_3(\boldsymbol{\kappa}, z) \end{pmatrix} \qquad \text{(6)}$$

Theory: ODE in K-Domain

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{V}}_{\perp} = \boldsymbol{f}(z, \tilde{\boldsymbol{V}}_{\perp})$$

$$\frac{d}{dz}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z) = \mathrm{i}k_0 \begin{bmatrix} 0 & 0 & \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} & 1 - \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ 0 & 0 & \frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} - 1 & -\frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ -\frac{k_x k_y}{k_0^2} & \frac{k_x^2}{k_0^2} - \underline{\tilde{\boldsymbol{\epsilon}}} & 0 & 0 \\ \underline{\tilde{\boldsymbol{\epsilon}}} - \frac{k_y^2}{k_0^2} & \frac{k_y k_x}{k_0^2} & 0 & 0 \end{bmatrix}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z) \quad (7)$$

$\underline{\tilde{\boldsymbol{\epsilon}}}$ and $\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}$ are the convolution operator. More specifically, $\underline{\tilde{\boldsymbol{\epsilon}}} = \tilde{\epsilon}*$ and $\underline{\tilde{\boldsymbol{\epsilon}}}^{-1} = \epsilon^{\tilde{-1}}*$

Mathematical task:
Solving the ordinary differential equation (ODE) (7), field propagation through media with $\breve{n}(\boldsymbol{r})$ is calculated!

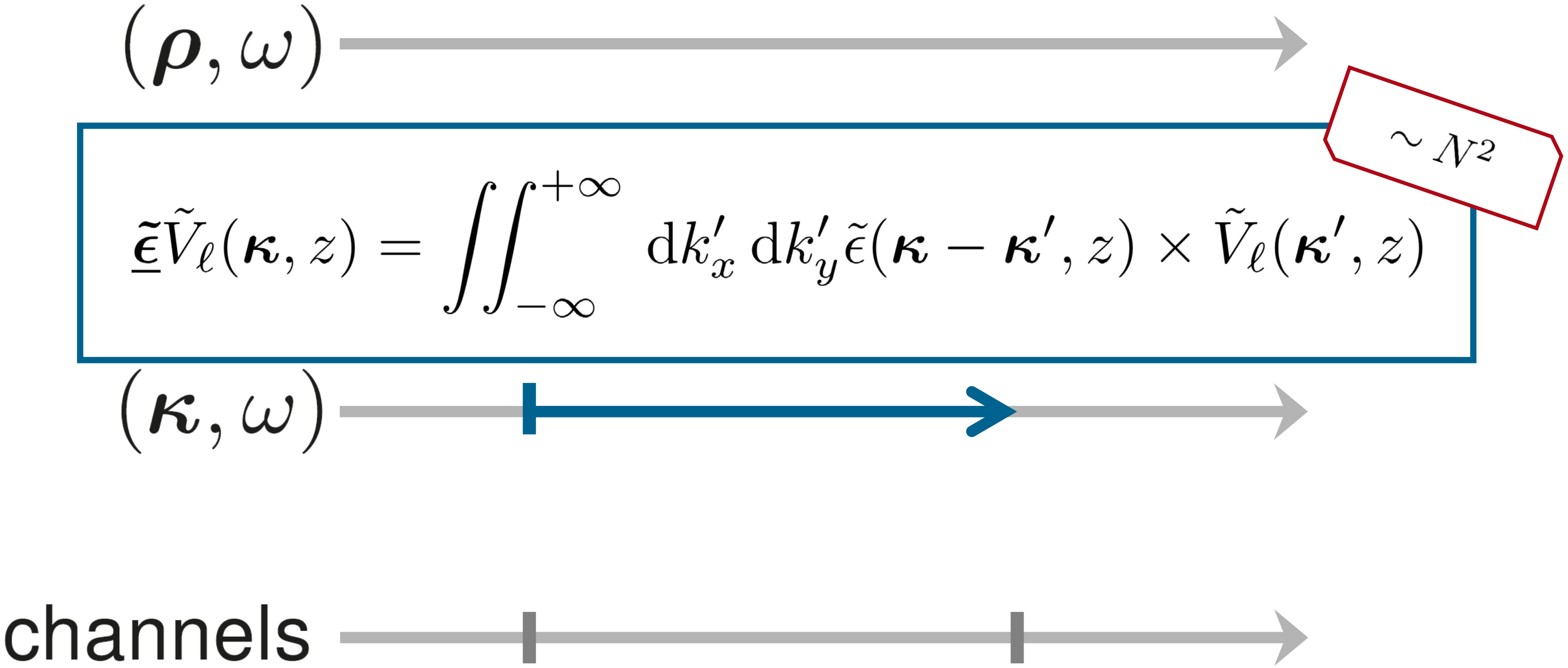
Theory: Solve the ODE

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{V}}_\perp = \boldsymbol{f}(z, \tilde{\boldsymbol{V}}_\perp)$$

$$\frac{d}{dz}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z) = \mathrm{i}k_0 \begin{bmatrix} 0 & 0 & \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} & 1-\frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ 0 & 0 & \frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0}-1 & -\frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ -\frac{k_x k_y}{k_0^2} & \frac{k_x^2}{k_0^2}-\underline{\tilde{\boldsymbol{\epsilon}}} & 0 & 0 \\ \underline{\tilde{\boldsymbol{\epsilon}}}-\frac{k_y^2}{k_0^2} & \frac{k_y k_x}{k_0^2} & 0 & 0 \end{bmatrix}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z)$$

How to deal with operator $\underline{\tilde{\boldsymbol{\epsilon}}}$ and $\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}$?

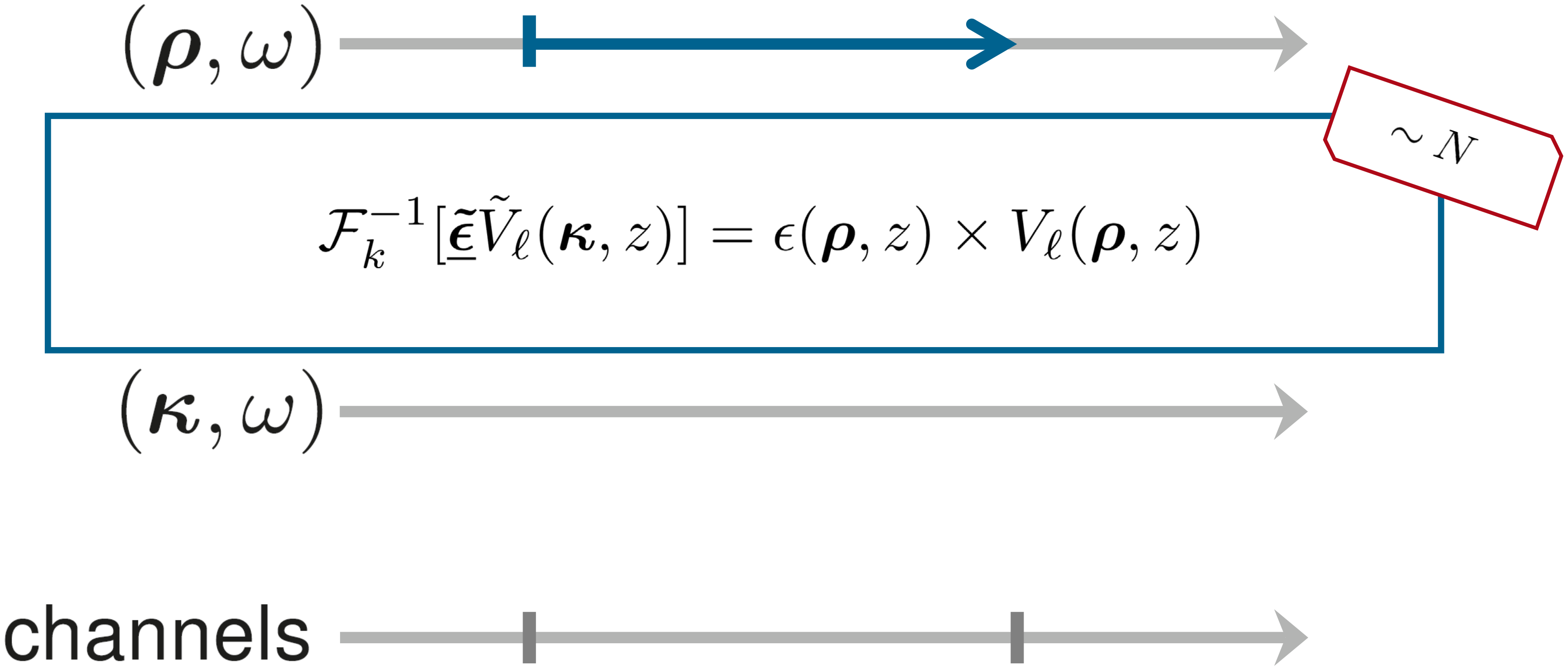
Theory: Convolution Operator



$$\underline{\tilde{\boldsymbol{\epsilon}}}\tilde{V}_\ell(\boldsymbol{\kappa}, z) = \iint_{-\infty}^{+\infty} \mathrm{d}k'_x \, \mathrm{d}k'_y \tilde{\epsilon}(\boldsymbol{\kappa} - \boldsymbol{\kappa}', z) \times \tilde{V}_\ell(\boldsymbol{\kappa}', z)$$

N is number of sampling points

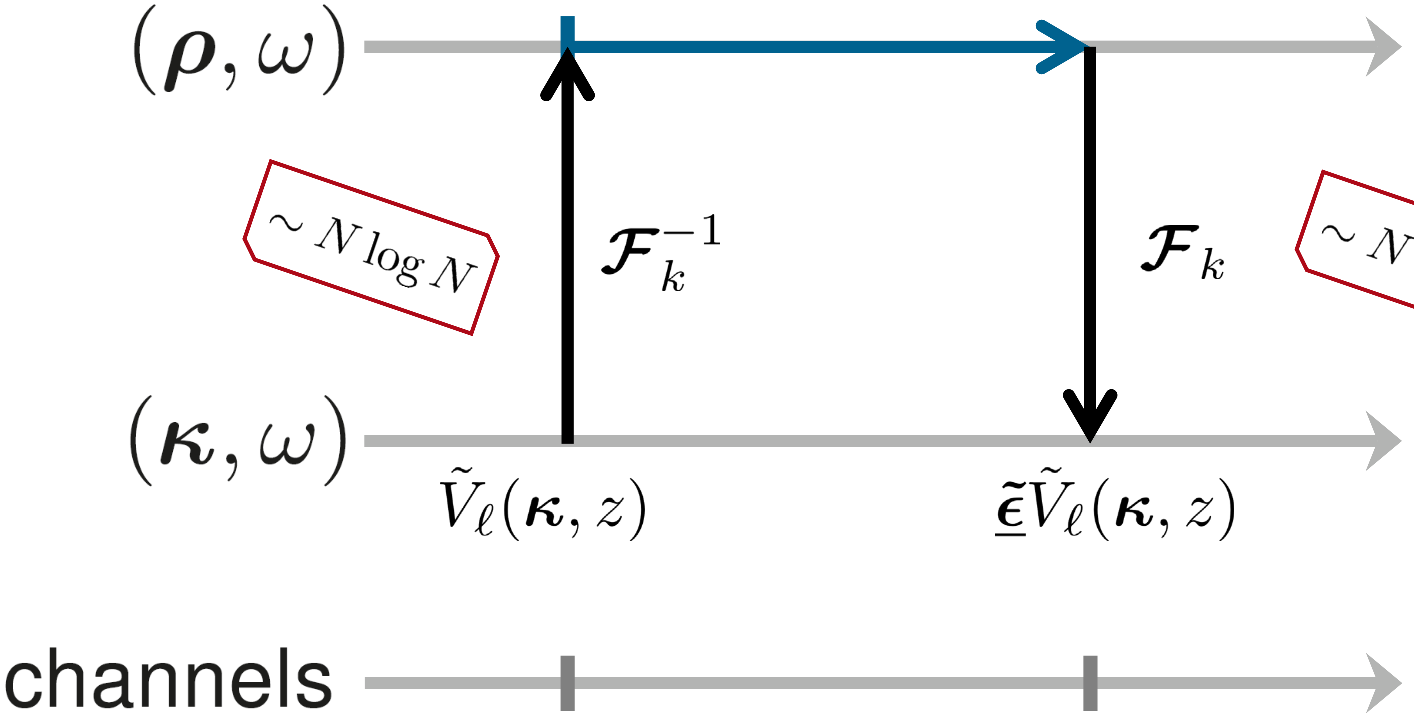
Theory: Convolution Theorem



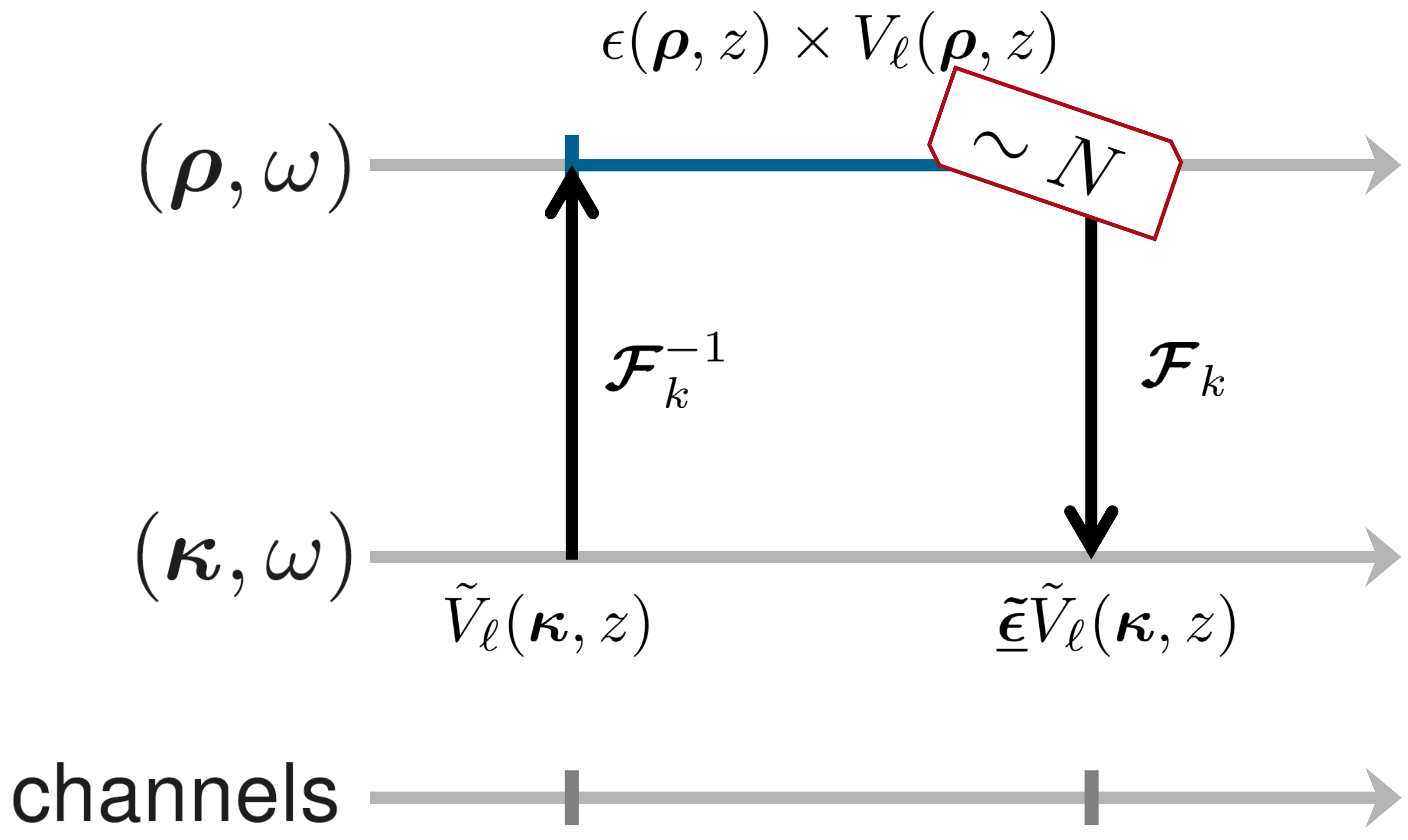
$$\mathcal{F}_k^{-1}[\tilde{\underline{\boldsymbol{\epsilon}}}\tilde{V}_\ell(\boldsymbol{\kappa}, z)] = \epsilon(\boldsymbol{\rho}, z) \times V_\ell(\boldsymbol{\rho}, z)$$

Theory: Convolution Operator

$$\mathcal{F}_k^{-1}[\underline{\tilde{\epsilon}}\tilde{V}_\ell(\boldsymbol{\kappa}, z)] = \epsilon(\boldsymbol{\rho}, z) \times V_\ell(\boldsymbol{\rho}, z)$$



Theory: Convolution Operator

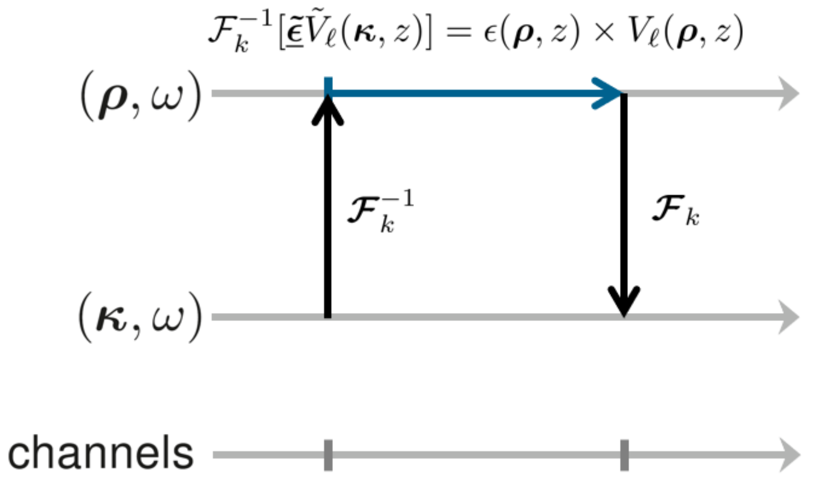


Theory: Convolution Operator

$$\frac{d}{dz}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa},z) = \mathrm{i}k_0\begin{bmatrix} 0 & 0 & \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} & 1-\frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ 0 & 0 & \frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0}-1 & -\frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ -\frac{k_xk_y}{k_0^2} & \frac{k_x^2}{k_0^2}-\underline{\tilde{\boldsymbol{\epsilon}}} & 0 & 0 \\ \underline{\tilde{\boldsymbol{\epsilon}}}-\frac{k_y^2}{k_0^2} & \frac{k_yk_x}{k_0^2} & 0 & 0 \end{bmatrix}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa},z)$$

$$\underline{\tilde{\boldsymbol{\epsilon}}}\tilde{V}_\ell(\boldsymbol{\kappa},z) = \mathcal{F}_k\left\{\epsilon(\boldsymbol{\rho},z)\times\mathcal{F}_k^{-1}\left[\tilde{V}_\ell(\boldsymbol{\kappa},z)\right]\right\}$$

$$\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\kappa_j\tilde{V}_\ell(\boldsymbol{\kappa},z) = \mathcal{F}_k\left\{\epsilon^{-1}(\boldsymbol{\rho},z)\times\mathcal{F}_k^{-1}\left[\kappa_j\tilde{V}_\ell(\boldsymbol{\kappa},z)\right]\right\}$$



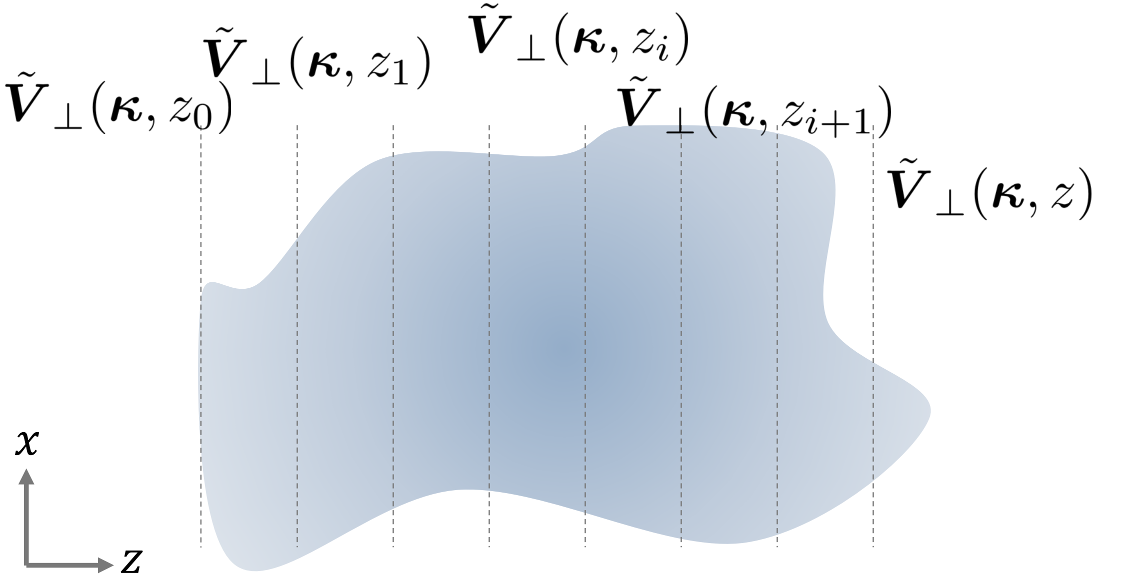
Theory: Solve the ODE

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{V}}_\perp = \boldsymbol{f}(z, \tilde{\boldsymbol{V}}_\perp)$$

$$\frac{d}{dz}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z) = \mathrm{i}k_0 \begin{bmatrix} 0 & 0 & \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} & 1-\frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ 0 & 0 & \frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0}-1 & -\frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ -\frac{k_x k_y}{k_0^2} & \frac{k_x^2}{k_0^2}-\underline{\tilde{\boldsymbol{\epsilon}}} & 0 & 0 \\ \underline{\tilde{\boldsymbol{\epsilon}}}-\frac{k_y^2}{k_0^2} & \frac{k_y k_x}{k_0^2} & 0 & 0 \end{bmatrix}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z)$$

ODE solver (initial value problem)

- Euler method
- Taylor series methods
- Runge-Kutta methods
- ...



Theory: Solve the ODE

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{V}}_{\perp}^{\mathrm{EM}} = \boldsymbol{f}(z, \tilde{\boldsymbol{V}}_{\perp}^{\mathrm{EM}})$$

$$\frac{d}{dz}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z) = \mathrm{i}k_0 \begin{bmatrix} 0 & 0 & \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} & 1 - \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ 0 & 0 & \frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} - 1 & -\frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ -\frac{k_x k_y}{k_0^2} & \frac{k_x^2}{k_0^2} - \underline{\tilde{\boldsymbol{\epsilon}}} & 0 & 0 \\ \underline{\tilde{\boldsymbol{\epsilon}}} - \frac{k_y^2}{k_0^2} & \frac{k_y k_x}{k_0^2} & 0 & 0 \end{bmatrix} \begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z)$$

ODE solver (initial value problem)

- Euler method
- Taylor series methods
- Runge-Kutta methods
- ...

Calculate $\tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_{i+1})$ from $\tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_i)$

$$\boldsymbol{k}_1 = \Delta z_i \boldsymbol{f}(z_i, \tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_i))$$

$$\boldsymbol{k}_2 = \Delta z_i \boldsymbol{f}(z_i + \frac{1}{2}\Delta z_i, \tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_i) + \frac{1}{2}\boldsymbol{k}_1)$$

$$\boldsymbol{k}_3 = \Delta z_i \boldsymbol{f}(z_i + \frac{1}{2}\Delta z_i, \tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_i) + \frac{1}{2}\boldsymbol{k}_2)$$

$$\boldsymbol{k}_4 = \Delta z_i \boldsymbol{f}(z_{i+1}, \tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_i) + \frac{1}{2}\boldsymbol{k}_3)$$

$$\tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_{i+1}) = \tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_i) + \frac{1}{6}(\boldsymbol{k}_1 + 2\boldsymbol{k}_2 + 2\boldsymbol{k}_3 + \boldsymbol{k}_4)$$

Theory: Solve the ODE

$$\frac{\mathrm{d}}{\mathrm{d}z}\tilde{\boldsymbol{V}}_{\perp}^{\mathrm{EM}} = \boldsymbol{f}(z, \tilde{\boldsymbol{V}}_{\perp}^{\mathrm{EM}})$$

$$\frac{d}{dz}\begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{V}_4 \\ \tilde{V}_5 \end{pmatrix}(\boldsymbol{\kappa}, z) = \mathrm{i}k_0 \begin{bmatrix} 0 & 0 & \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} & 1 - \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ 0 & 0 & \frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} - 1 & -\frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ -\frac{k_x k_y}{k_0^2} & \frac{k_x^2}{k_0^2} - \underline{\tilde{\boldsymbol{\epsilon}}} & 0 & 0 \\ \underline{\tilde{\boldsymbol{\epsilon}}} - \frac{k_y^2}{k_0^2} & \frac{k_y k_x}{k_0^2} & 0 & 0 \end{bmatrix} \begin{pmatrix} \tilde{V}_1 \\ \tilde{V}_2 \\ \tilde{V}_4 \\ \tilde{V}_5 \end{pmatrix}(\boldsymbol{\kappa}, z)$$

ODE solver (initial value problem)

- Euler method
- Taylor series methods

Calculate $\tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_{i+1})$ from $\tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_i)$

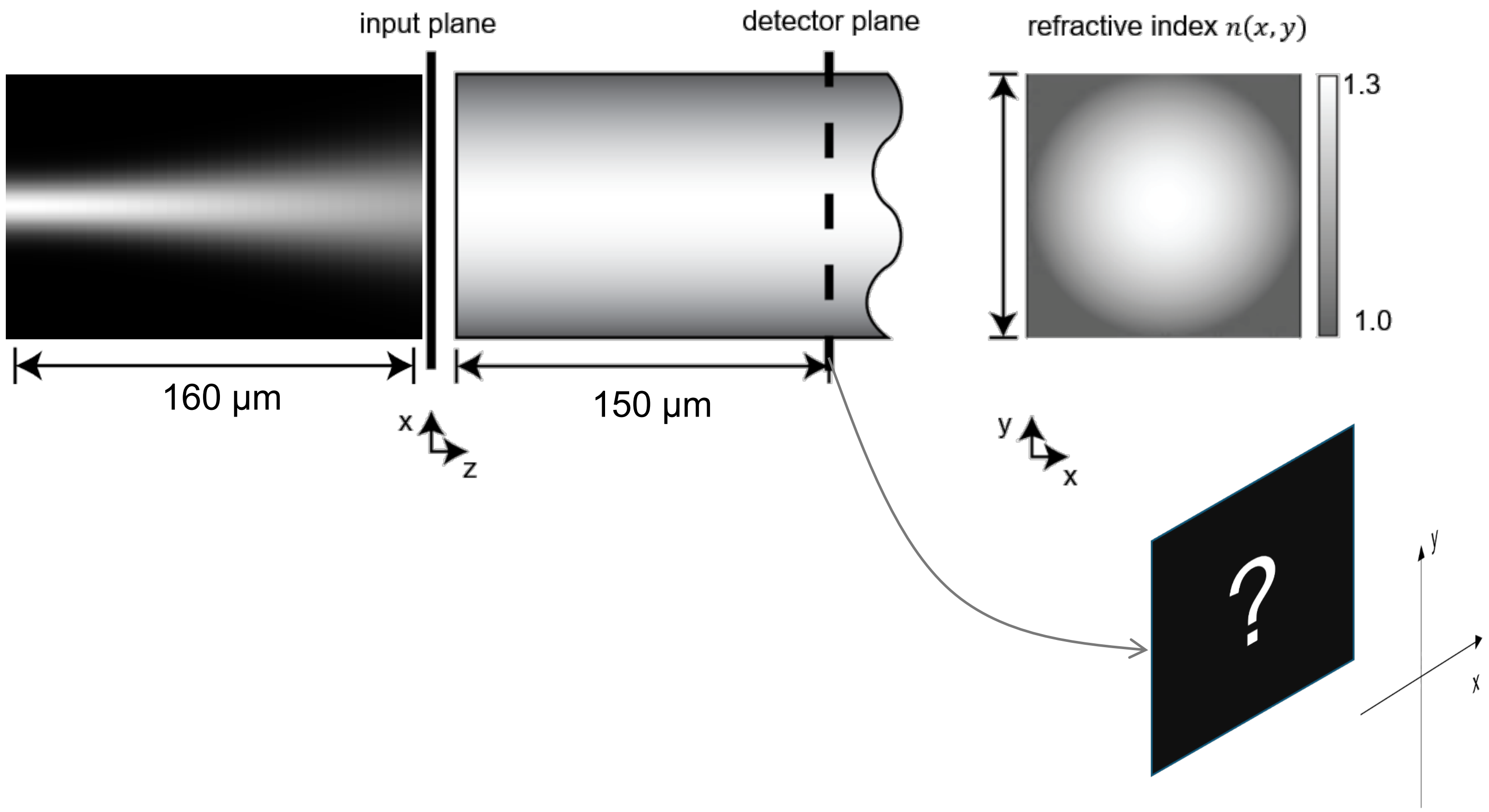
$$\boldsymbol{k}_1 = \Delta z_i \boldsymbol{f}(z_i, \tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_i))$$

$$\boldsymbol{k}_2 = \Delta z_i \boldsymbol{f}(z_i + \frac{1}{2}\Delta z_i, \tilde{\boldsymbol{V}}_{\perp}(\boldsymbol{\kappa}, z_i) + \frac{1}{2}\boldsymbol{k}_1)$$

We name the $k-$domain method as Runge-Kutta $k-$domain algorithm.

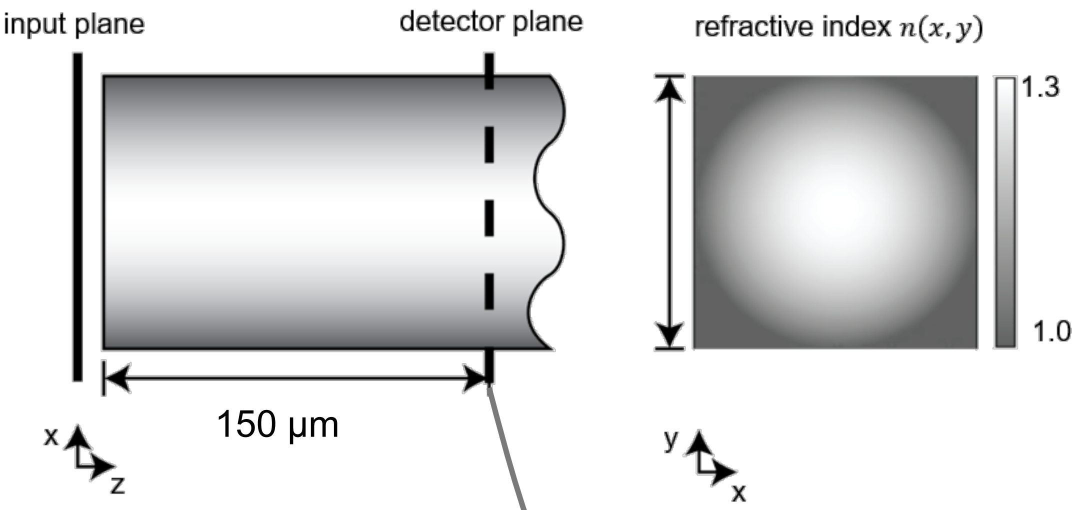
Example: Multimode Fiber

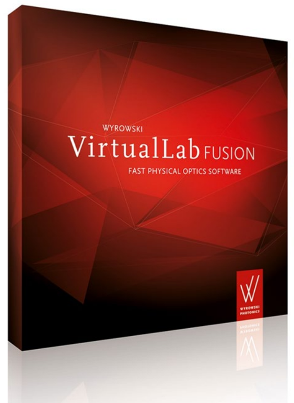
Input Gaussian
- waist radius 5 µm
- Ey-polarized
- wavelength 532 nm



calculate the result fields by Fourier modal method and Runge-Kutta based k-domain algorithm.

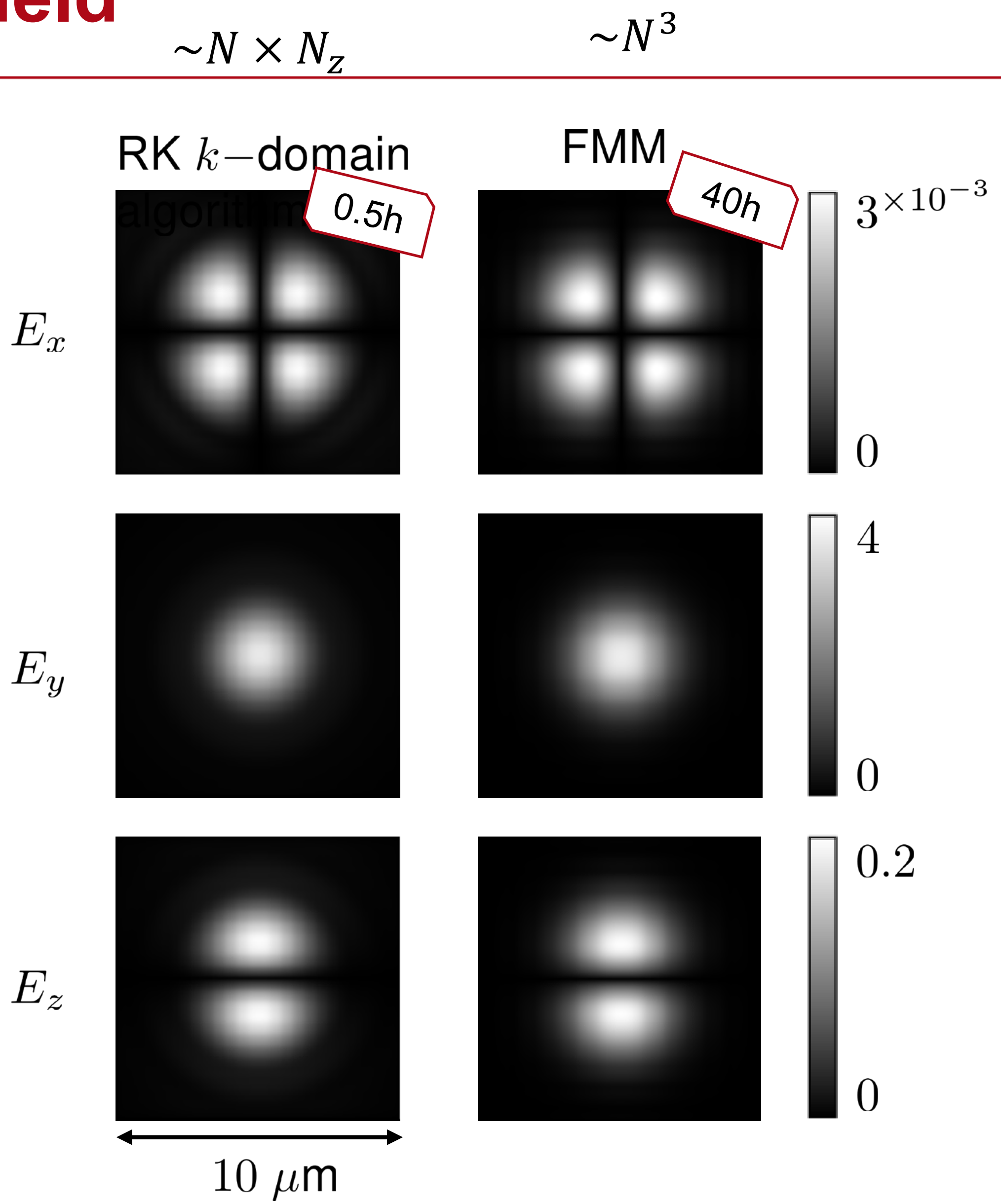
Result: Amplitude [V/m] of Output Field





Implemented by using *Programmable Component* in VLF

Deviation ~1%



Two-Dimentional Case

Theory: ODE for y –Invariant Condition

$\partial_y = 0$

$$\frac{d}{dz}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z) = ik_0 \begin{bmatrix} 0 & 0 & 0 & 1-\frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ 0 & 0 & -1 & 0 \\ 0 & \frac{k_x^2}{k_0^2}-\underline{\tilde{\boldsymbol{\epsilon}}} & 0 & 0 \\ \underline{\tilde{\boldsymbol{\epsilon}}} & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z) \quad (8)$$

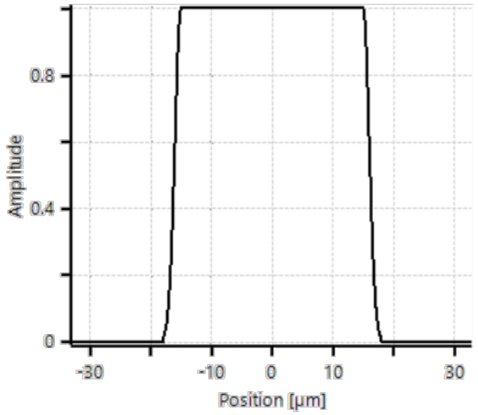
TE

$$\frac{d}{dz}\begin{pmatrix}\tilde{V}_2\\ \tilde{V}_4\end{pmatrix}(\boldsymbol{\kappa}, z) = ik_0 \begin{bmatrix} 0 & -1 \\ \frac{k_x^2}{k_0^2}-\underline{\tilde{\boldsymbol{\epsilon}}} & 0 \end{bmatrix}\begin{pmatrix}\tilde{V}_2\\ \tilde{V}_4\end{pmatrix}(\boldsymbol{\kappa}, z) \quad (9)$$

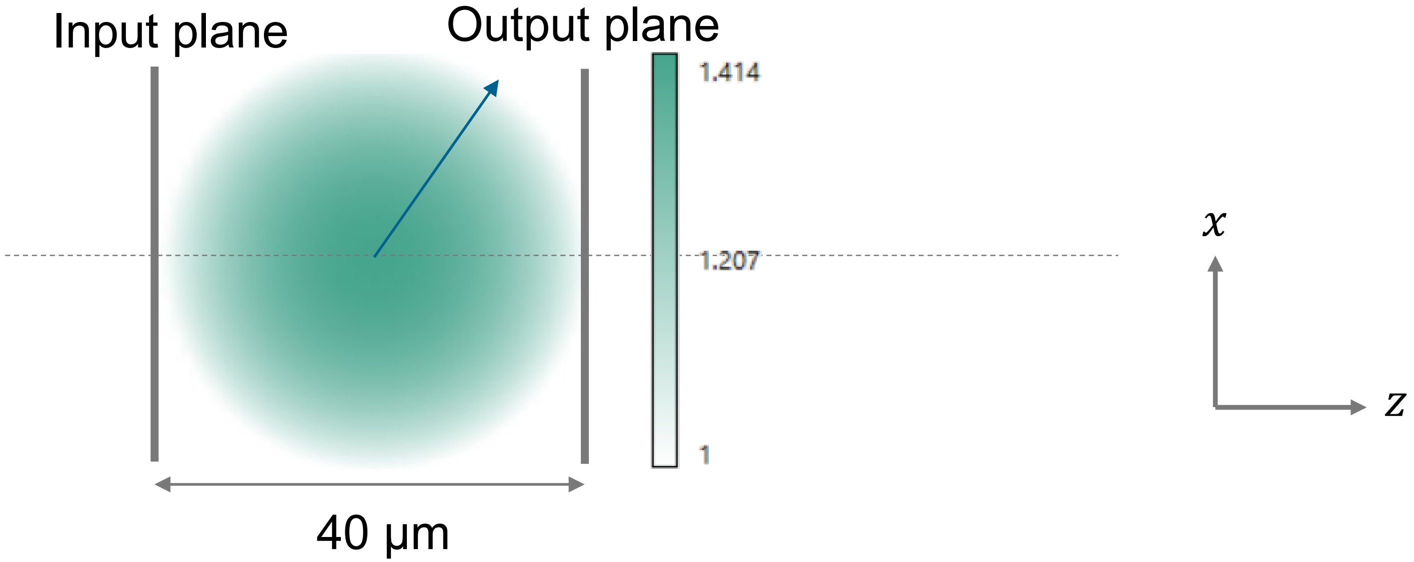
TM

$$\frac{d}{dz}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z) = ik_0 \begin{bmatrix} 0 & 1-\frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0} \\ \underline{\tilde{\boldsymbol{\epsilon}}} & 0 \end{bmatrix}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa}, z) \quad (10)$$

Y-Invariant GRIN Media: Luneburg Cylinder Lens



Input plane wave
- beam size 38 µm
- E_x-polarized
- wavelength 532 nm



GRIN media:

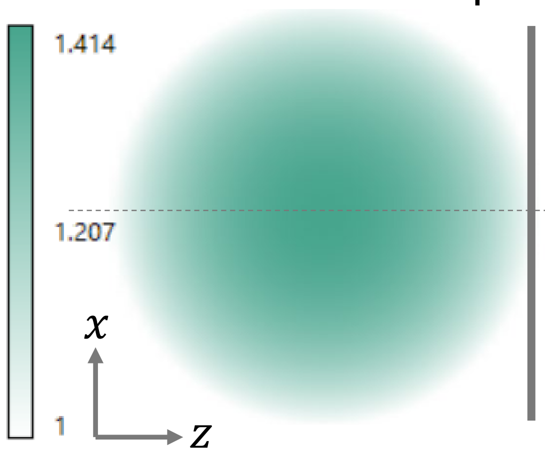
-$n(x) = \sqrt{2 - (x^2 + z^2)/R^2}$
with $R = 20\ \mu$m

Task: By using FMM (rigorous) and the RK k-domain algorithm

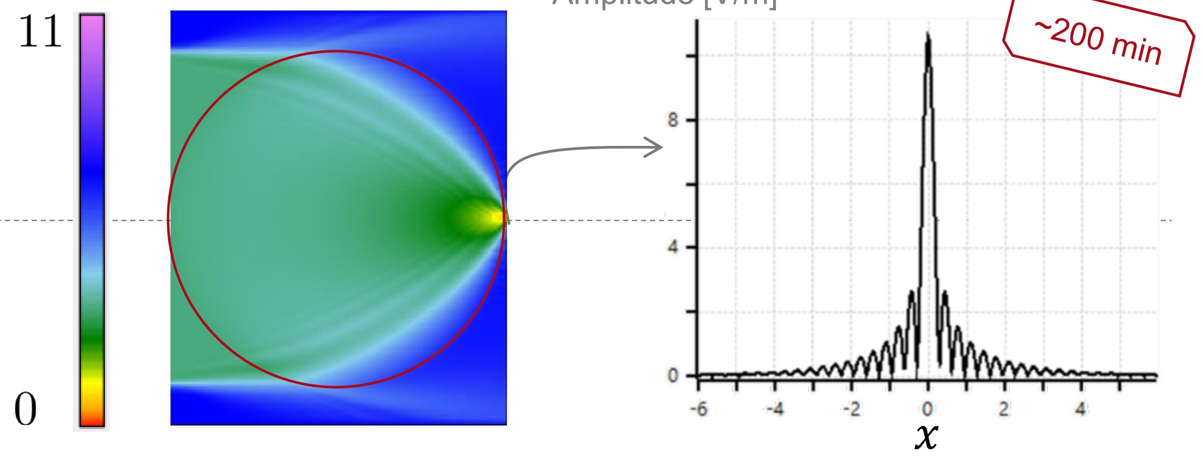
- calculate field propagation in GRIN media xz −plane
- calculate field in the output plane

Result: Amplitude of E_x –Field

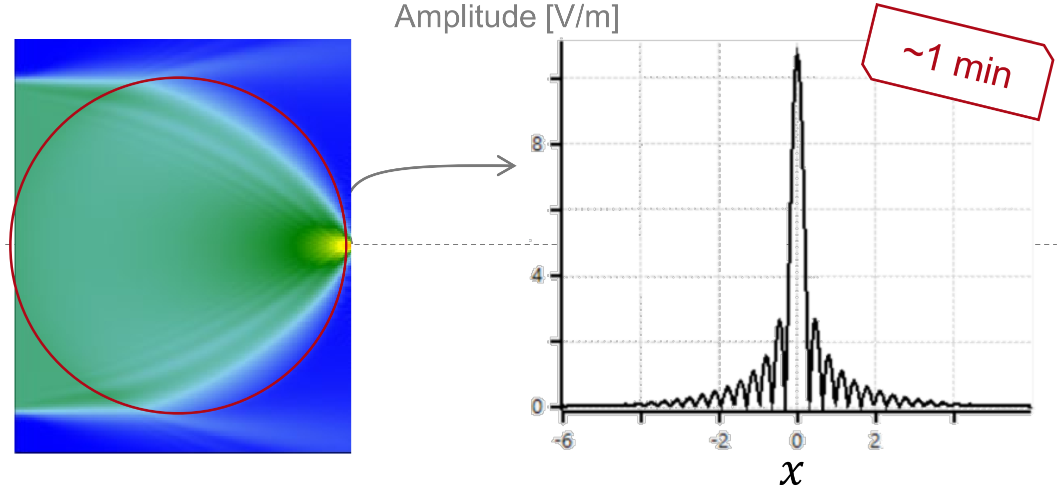
Output plane



FMM

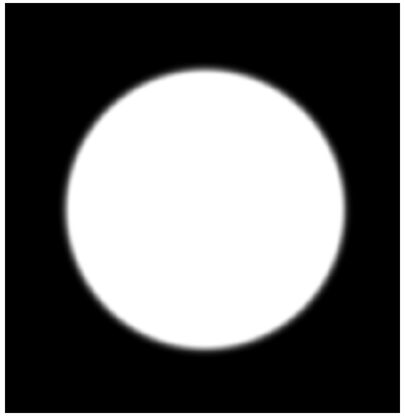


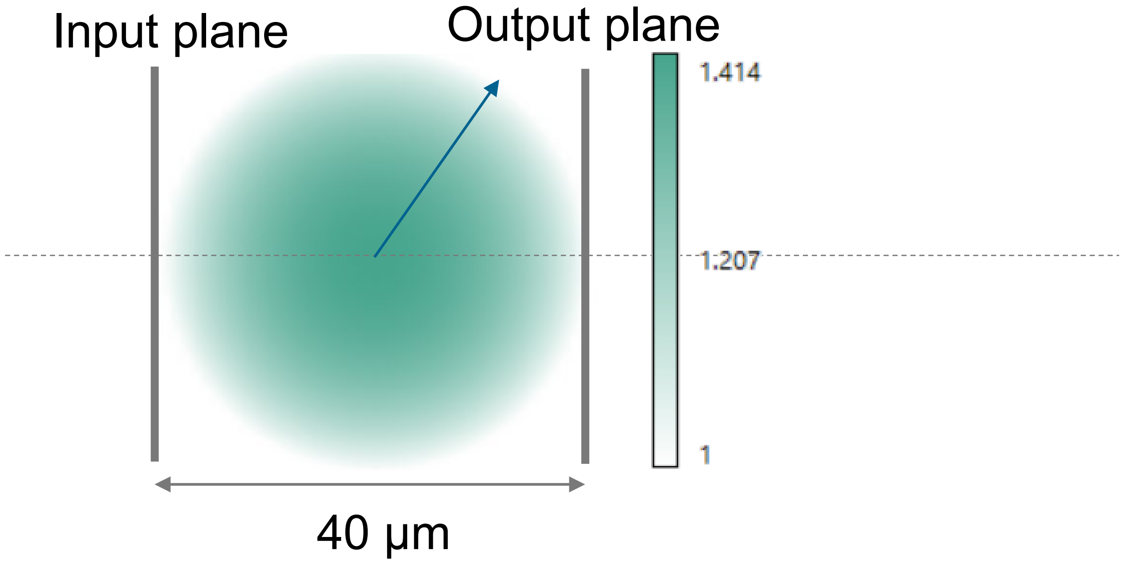
RK k-domain algorithm

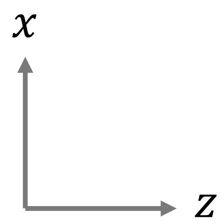


NA at focus position > 0.7

3D Case: Luneburg Lens





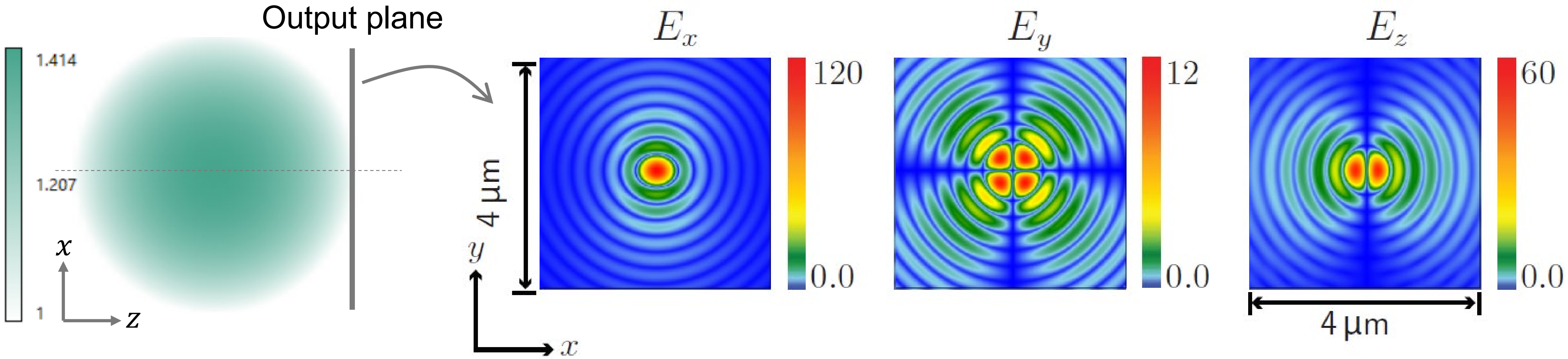


Input plane wave
- beam size 38 µm
- E_x-polarized
- wavelength 532 nm

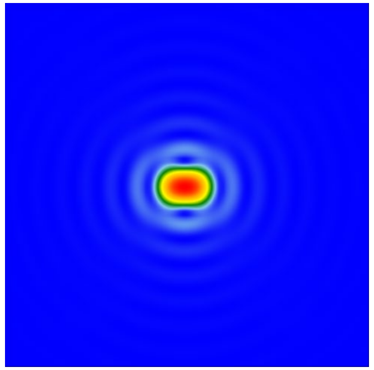
GRIN media:
-$n(x) = \sqrt{2 - (x^2 + y^2 + z^2)/R^2}$
with $R = 20\ \mu$m

Task: By using FMM (rigorous) and the RK k-domain algorithm
– calculate field in the output plane

Result: Amplitude and Energy Density of Electric Fields



energy density



$$\sim |E_x|^2 + |E_y|^2 + |E_z|^2$$

Conclusion

- Develop a fast k-domain algorithm to calculate field propagation through graded-index media
  - Maxwell's equations to derive ODE

$$\frac{d}{dz}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa},z)=\mathrm{i}k_0\begin{bmatrix}0 & 0 & \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} & 1-\frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0}\\ 0 & 0 & \frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0}-1 & -\frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0}\\ -\frac{k_x k_y}{k_0^2} & \frac{k_x^2}{k_0^2}-\underline{\tilde{\boldsymbol{\epsilon}}} & 0 & 0\\ \underline{\tilde{\boldsymbol{\epsilon}}}-\frac{k_y^2}{k_0^2} & \frac{k_y k_x}{k_0^2} & 0 & 0\end{bmatrix}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa},z)$$

  - Solving this ODE by Runge-Kutta method (4th order) slice by slice along z −axis
  - By using convolution theorem, convolution in k-domain is realized by multiplication in spatial domain. So numerical effort of this algorithm $\sim N \times N_z$, with N is sampling points of field and N_z denoting slice number
- Compared to FMM the RK k-domain algorithm shows advantage when N becomes large, which is general in three dimentional cases; it has no limit in $\check{n}(x,y,z)$ and NA of field.

Outlook: Further Tricks of Solver

- We rewrite $\tilde{\boldsymbol{V}}_\perp = \tilde{\boldsymbol{U}}_\perp \exp(\mathrm{i}k_0\bar{n}z)$, which abstract the fast changing term of field, ODE becomes

$$\frac{d}{dz}\begin{pmatrix}\tilde{U}_1\\ \tilde{U}_2\\ \tilde{U}_4\\ \tilde{U}_5\end{pmatrix}(\boldsymbol{\kappa}, z) = \mathrm{i}k_0\begin{bmatrix} -\bar{n} & 0 & \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} & 1-\frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0}\\ 0 & -\bar{n} & \frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0}-1 & -\frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0}\\ -\frac{k_x k_y}{k_0^2} & \frac{k_x^2}{k_0^2}-\underline{\tilde{\boldsymbol{\epsilon}}} & -\bar{n} & 0\\ \underline{\tilde{\boldsymbol{\epsilon}}}-\frac{k_y^2}{k_0^2} & \frac{k_y k_x}{k_0^2} & 0 & -\bar{n}\end{bmatrix}\begin{pmatrix}\tilde{U}_1\\ \tilde{U}_2\\ \tilde{U}_4\\ \tilde{U}_5\end{pmatrix}(\boldsymbol{\kappa}, z)$$

  Slow varying term $\boldsymbol{U}_\perp$ is calculated, so N_z can be reduced

- In general case, $\tilde{\boldsymbol{V}}_\perp = \tilde{\boldsymbol{U}}_\perp \exp(\mathrm{i}\tilde{\phi})$ or $\boldsymbol{V}_\perp = \boldsymbol{U}_\perp \exp(\mathrm{i}\psi)$. We need to explore how to predict $\tilde{\psi}$ or ψ and how to perform Fourier transform fast!

Related Talks and Poster Presentations

- **Talk**: Optimization of coupling gratings for lightguide-based displays
  Time: Monday, 1 April 2019 | 16:00 – 17:00

- **Poster**: Modeling of Diffractive/Meta-Lenses using Fast Physical Optics
  Time: Monday, 1 April 2019 | 10:55 – 11:25
- **Poster**: Vectorial physical-optics modeling of the interaction of a tightly focused beam with a nanoparticle
  Time: Monday, 1 April 2019 | 10:55 – 11:25

Thank you!

Discussion about Numerical Effort

- The calculation is along $z-$axis slice by slice. Assume the slice number is N_z.
- In each slice, RK4 is used, which means we calculate four time $\boldsymbol{f}(z, \tilde{\boldsymbol{V}}_\perp)$ with operations $\sim N$. N is number of sampling points of field

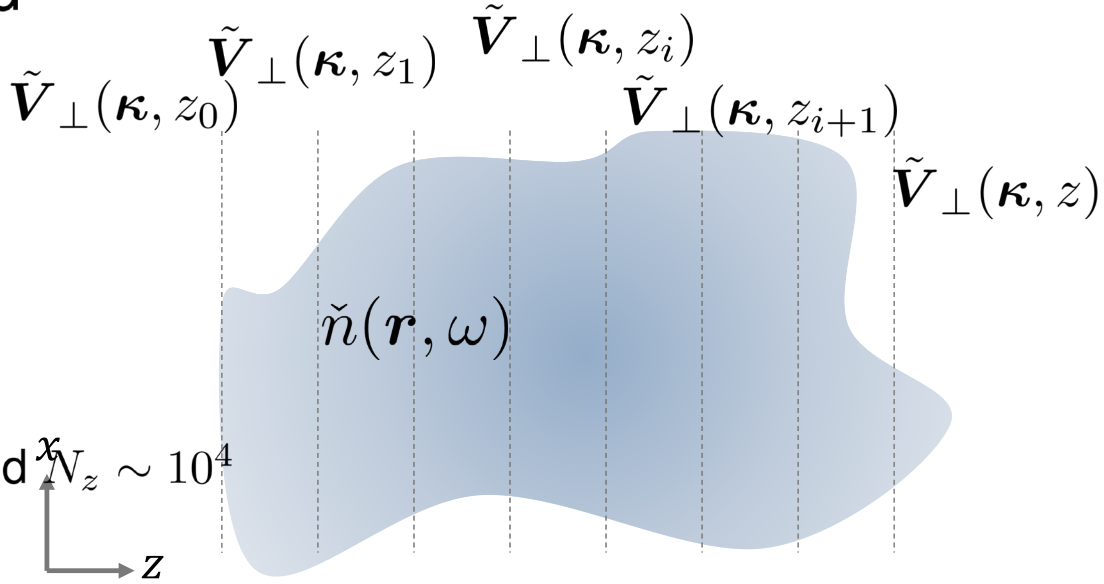
The numerical effort of Runge Kutta based $k-$domain method is $\sim N \times N_z$

FMM ($\sim N^3 \times N_z$)

In this example, $N \sim 10^4$

- FMM $N^3 \sim 10^{12}$ and $N_z = 1$
- Runge-Kutta based k-domain algorithm $N \sim 10^4$ and $N_z \sim 10^4$

Advantage of fast calculation when N becomes large, typically in 3D



Theory: Convolution Operator

$$\frac{d}{dz}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa},z) = \mathrm{i}k_0\begin{bmatrix} 0 & 0 & \frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0} & 1-\frac{k_x}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0}\\ 0 & 0 & \frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_y}{k_0}-1 & -\frac{k_y}{k_0}\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}\frac{k_x}{k_0}\\ -\frac{k_xk_y}{k_0^2} & \frac{k_x^2}{k_0^2}-\underline{\tilde{\boldsymbol{\epsilon}}} & 0 & 0\\ \underline{\tilde{\boldsymbol{\epsilon}}}-\frac{k_y^2}{k_0^2} & \frac{k_yk_x}{k_0^2} & 0 & 0\end{bmatrix}\begin{pmatrix}\tilde{V}_1\\ \tilde{V}_2\\ \tilde{V}_4\\ \tilde{V}_5\end{pmatrix}(\boldsymbol{\kappa},z)$$

How to deal with operator $\underline{\tilde{\boldsymbol{\epsilon}}}$ and $\underline{\tilde{\boldsymbol{\epsilon}}}^{-1}$? Instead of convolution (N^2) in $k-$domain, (inverse) Fourier transform and multiplication in spatial domain is used.

$$\mathcal{F}_k^{-1}[\underline{\tilde{\boldsymbol{\epsilon}}}\tilde{V}_\ell(\boldsymbol{\kappa},z)] = \epsilon(\boldsymbol{\rho},z)\times V_\ell(\boldsymbol{\rho},z)$$